Uniform Convergence

Where is it prove that one obtains the derivative of an infinite series by taking derivative of each term ? - - -Niels Henrik Abel

We have already studied sequences and series of (constant) real numbers. In most problems, however, it is desirable to approximate functions by more elementary ones that are easier to investigate. We have already done this on a few occasions. For example, we looked at the uniform approximation of continuous functions by step, piecewise linear, and polynomial functions. Also, we proved that each bounded continuous function on a closed bounded interval is a uniform limit of regulated functions. Now all these approximations involve estimates on the distance between the given continuous function and the elementary functions that approximate it. This in turn suggests the introduction of sequences (and hence also series) whose terms are functions defined, in most cases, on the same interval.

1 Sequence and Series of Functions

(i) For a set $E \subset \mathbb{R}$, let us denote by $\mathcal{F}(E;\mathbb{R})$ the set of all functions from E to $\mathbb{R}$. We are interested in sequences and series in the sets $\mathcal{F}(E;\mathbb{R})$. For each sequence $\langle f_n \rangle \in \mathcal{F}(E;\mathbb{R})^{\mathbb{N}}$ and each $x \in E$, the numerical sequence $\langle f_n(x) \rangle \in R^{\mathbb{N}}$ may or may not converge. Let $f_n : [0,1] \to \mathbb{R}$ be a function, given by $f_n(x) = \frac{x}{n}; x \in [0,1]$, then $\langle f_n \rangle$ is a sequence of functions on $[0,1]$.

(ii) Let $E \subset \mathbb{R}$ and let $\langle u_n \rangle \in \mathcal{F}(E;\mathbb{R})^{\mathbb{N}}$. Then the formal sum

$$u_1 + u_2 + \cdots + u_n + \cdots = \sum_{n=1}^{\infty} u_n$$

is called an *infinite series of functions* with general term u_n. For each $x \in E$, we have a numerical series

$$u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots = \sum_{n=1}^{\infty} u_n(x)$$

For each $n \in \mathbb{N}$, we can then define the partial sum

$$s_n(x) = u_1(x) + u_2(x) + \cdots + u_k(x) = \sum_{k=1}^{n} u_k(x)$$

This defines a sequence $\langle s_n \rangle \in \mathcal{F}(E;\mathbb{R})^{\mathbb{N}}$.

2 Pointwise Convergence

Definition 1. [Pointwise Convergent Sequence of functions]: For each $\langle f_n \rangle \in \mathcal{F}(E;\mathbb{R})^{\mathbb{N}}$; let $E_0 \subset E$ be the set of all points $x \in E$ such that the numerical sequence $\langle f_n(x) \rangle \in \mathbb{R}^{\mathbb{N}}$ converges and let

$$f(x) = \lim_{n \longrightarrow \infty} f_n(x); \qquad \forall x \in E_0 \tag{1}$$

which, in detail, means that, corresponding to an $\varepsilon > 0$, $\exists N = N(x;\varepsilon) \in \mathbb{N}$, depends on both x and ε, such that

$$|f_n(x) - f(x)| < \varepsilon; \quad \text{whenever } n \geq N \tag{2}$$

The sequence $\langle f_n \rangle$ is then said to be *pointwise convergent* (or simply convergent) on E_0 and the function $f \in \mathcal{F}(E_0;\mathbb{R})$; defined by (1) is called the *pointwise limit* (or simply limit) of $\langle f_n \rangle$ on E_0.

For example, let $X = \{1, 2, 3\}$ and let $f_n(k) \equiv n(mod k)$; $k = 1, 2, 3$ where $n(mod k)$ is the remainder when n is divided by k. Let $a = 1$, then $f_n(1) = 0$ for $n \in \mathbb{N}$ and hence $f_n(1) \to 0$. On the other hand $\langle f_n(2) \rangle = \langle 1, 0, 1, 0, \cdots \rangle$ and hence the sequence is not convergent. Hence the sequence $\langle f_n \rangle$ is not pointwise convergent on X.

We now look at a few examples and examine their pointwise convergence. Pay attention to the graphs of these functions to get an idea of what is going on. As far as possible, we shall investigate whether the given sequence is pointwise convergent and if so, we shall determine the limit function.

EXAMPLE **1.** (A discontinuous limit of continuous functions) Consider, the sequence $\langle f_n(x) \rangle_n$, where, $f_n(x) = x^n$, for all $x \in [0; 1]$ as depicted in the Fig. 1. For $x \in [0, 1)$. We then have $\lim_{n \longrightarrow \infty} f_n(x) = 0$. On the other hand, $\lim_{n \longrightarrow \infty} f_n(1) = 1$. The sequence is therefore pointwise convergent to a function $f(x)$, on $[0, 1]$, where

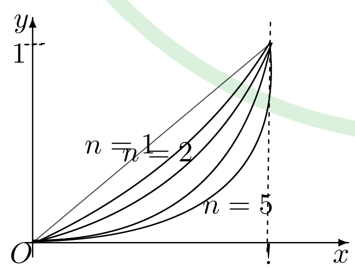

Figure 1: $f_n(x)$ for $n = 1, 2, 3, 4, 5$.

$$f(x) = \begin{cases} 0; & \text{if } 0 \leq x < 1, \\ 1; & \text{if } x = 1 \end{cases}$$

Note that each function $f_n(x)$ of the sequence is continuous on $[0, 1]$, but the limit function is not continuous on $[0, 1]$, it has a jump discontinuity at the point $x = 1$.

EXAMPLE **2.** Consider, the sequence $\langle f_n(x) \rangle_n$, where, $f_n(x) = \left(1 - \frac{nx}{n+1}\right)^{n/2}$; $n \geq 1$ for all $x \in (-\infty, 1]$. We then have $\lim_{n \longrightarrow \infty} f_n(x) = 0$, for $0 < x < 1$ and $\lim_{n \longrightarrow \infty} f_n(x) = \infty$, for $x < 0$. On the other hand, $\lim_{n \longrightarrow \infty} f_n(0) = 1$. Thus, the sequence $\langle f_n(x) \rangle_n$ converges pointwise on $E_0 = [0, 1]$ to the limit function f defined by

$$f(x) = \begin{cases} 0; & \text{if } 0 \leq x \leq 1, \\ 1; & \text{if } x = 0 \end{cases}$$

EXAMPLE **3.** Consider, the sequence $\langle f_n(x)\rangle_n$, where, $f_n(x) = x^n e^{-nx}; n \geq 1$ and $x \geq 0$, as

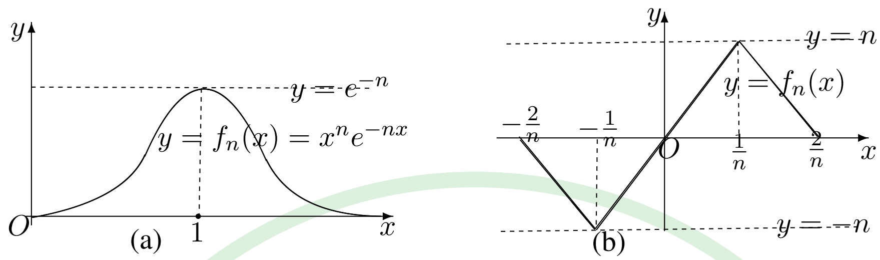

Figure 2: (a) $f_n(x) = x^n e^{-nx}$ (b) $f_n(x)$.

depicted in the Fig. 2(a). Now, $f_n'(x) = nx^{n-1}e^{-nx}(1-x) = 0$, gives $x = 1$ and the maximum value of $f_n(x)$ on $[0,\infty)$ is e^{-n}. Therefore $|f_n(x)| \leq e^{-n}$, and so $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \geq 0$. The limit function in this case is identically zero on $[0,\infty)$.

EXAMPLE **4.** Consider, the sequence $\langle f_n(x)\rangle_n$, where, for $n \geq$,

$$f_n(x) = \begin{cases} 0; & \text{for } x < -\frac{2}{n} \\ -n(2+nx); & \text{for } -\frac{2}{n} \leq x < -\frac{1}{n} \\ n^2x; & \text{for } -\frac{1}{n} \leq x < \frac{1}{n} \\ n(2-nx); & \text{for } \frac{1}{n} \leq x < \frac{2}{n} \\ 0; & \text{for } x \geq \frac{2}{n} \end{cases}$$

defined on $(-\infty,\infty)$, as depicted in the Fig. 2(b). Here $\lim\limits_{n\to\infty} f_n(0) = 0$, for all n and $\lim\limits_{n\to\infty} f_n(x) = 0$ if $n \geq \frac{2}{|x|}$. Therefore

$$f(x) = \lim_{n\to\infty} f_n(x) = 0; \;\; -\infty < x < \infty,$$

so, the limit function in this case is identically zero on $(-\infty,\infty)$.

EXAMPLE **5.** (Uniform limits of differentiable functions need not be differentiable): Consider,

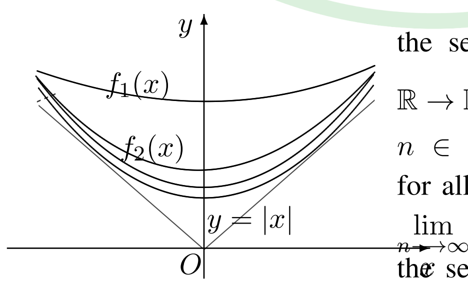

Figure 3: Uniform limits of differentiable functions need not be differentiable

the sequence $\langle f_n(x)\rangle_n$, where $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$, $n \in \mathbb{N}$, as depicted in the Figure 3, for all $x \in \mathbb{R}$. Here we clearly have $\lim\limits_{n\to\infty} f_n(x) = |x|$ for all $x \in \mathbb{R}$. Thus, the sequence is pointwise convergent on $\mathbb{R}$. We also observe that f_n is differentiable on $\mathbb{R}$ for all $n \in \mathbb{N}$ with $f_n'(x) = \dfrac{x}{\sqrt{x^2 + \frac{1}{n}}}$.

On the other hand, the limit function $f(x) = |x|$ is not differentiable at $x = 0$.

EXAMPLE **6.** Consider, the sequence $\langle f_n(x)\rangle_n$, where, $f_n(x) = \dfrac{\sin(n^2x)}{n}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

as depicted in the Figure 6. Since $|\sin \alpha| \leq 1$ for all $\alpha \in \mathbb{R}$, we have obvious estimate $|f_n(x)| \leq \frac{1}{n}$ for any $x \in \mathbb{R}$. Here the limit function, f; is the (identically) zero function. Indeed, $\left|\dfrac{\sin(n^2x)}{n}\right| \leq \dfrac{1}{n}$ holds for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$ and $\lim\limits_{n\longrightarrow\infty} \frac{1}{n} = 0$.

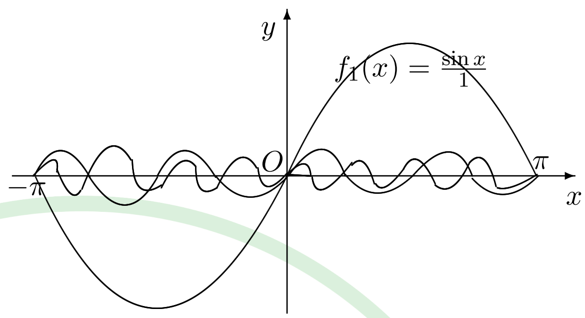

Figure 4: Graph of $f_n(x)$ of Example 6

Therefore, $f(x) = f'(x) = 0$ for all $x \in \mathbb{R}$. On the other hand,

$$f_n'(x) = n^2 \cdot \frac{\cos(n^2x)}{n} = n\cos(n^2x)$$

does not converge to 0. In fact, $\lim\limits_{n\longrightarrow\infty} f_n'(0) = \lim\limits_{n\longrightarrow\infty}(n) = \infty$.

EXAMPLE **7.** Consider, the sequence $\langle f_n(x)\rangle_n$, where, $f_n(x) = [\cos^2(n!\pi x)]$, for all $x \in [0,1]$ where, for each $t \in \mathbb{R}$, $[t]$ denotes the greatest integer $\leq t$. If $x = \frac{p}{q}$ with (relatively prime) positive integers p and q; then $n!x$ is an integer for all $n \geq q$ and hence $\cos^2(n!\pi x) = 1$. On the other hand, if $x \in \mathbb{Q}$, then $\cos^2(n!\pi x) \in (0,1)$. It follows that the (pointwise) limit function, f, is given by

$$f(x) = \begin{cases} 0; & \text{if } x \in \mathbb{Q} \cap [0,1], \\ 1; & \text{if } x \in [0,1]/\mathbb{Q} \end{cases}$$

In other words, f is the Dirichlet function which is nowhere continuous on $[0,1]$. In particular, f is not Riemann integrable. On the other hand, each f_n has only a finite (in fact $n!+1$) number of discontinuity points and hence is Riemann integrable.

EXAMPLE **8.** Consider the functions $f_n(x) = nx(1-x^2)^n$, for all $x \in [0,1]$. The (pointwise) limit, f; is the identically zero function: $f(x) = 0; \forall x \in [0,1]$. This is obvious for $x = 0$ and $x = 1$, and for $x \in (0,1)$ it follows from the fact that $\lim\limits_{n\longrightarrow\infty} n\alpha^n = 0$, for all $\alpha \in (0,1)$. Now

$$\int_0^1 f_n(x)dx = -\frac{n}{2}\Big[\frac{x(1-x^2)^{n+1}}{n+1}\Big]_0^1 = \frac{n}{2(n+1)}$$

It follows that $\lim\limits_{n\longrightarrow\infty} \displaystyle\int_0^1 f_n(x)dx = \frac{1}{2}$; and yet $\int_0^1 f(x)dx = 0$.

EXAMPLE **9.** For each positive integer $n \in \mathbb{N}$, let E_n be the set of numbers of the form $x = \frac{p}{q}$, where p and q are integers with no common factors and $1 \leq q \leq n$. Define $f_n(x) = \begin{cases} 1; & x \in E_n \\ 0; & x \notin E_n \end{cases}$. If x is irrational, then $x \notin E_n$, for any n, so $f_n(x) = 0$, $n \geq 1$. If x is rational, then $x \in E_n$ and $f_n(x) = 1$ for all sufficiently large n. Therefore

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 1; & x \in \mathbb{Q} \\ 0; & x \notin \mathbb{Q}^c = \mathbb{R}/\mathbb{Q} \end{cases}$$

EXAMPLE **10.** Let $q_l, q_2, ...$ be an enumeration of the rationals $\mathbb{Q} \cap [0,1]$ in the interval $I = [0,1]$. Consider the functions $f_n[0,1] \to \mathbb{R}$ defined by:

$$f_n(x) = \begin{cases} 1; & \text{if } x \in \{q_l, q_2, \dots, q_n\} \\ 0; & otherwise \end{cases}$$

Then the functions f_3 converge pointwise to the f which is equal to 1 on the rationals and 0 on the irrationals. Each f_n is integrable because it is discontinuous at only a finite number of points. But the pointwise limit is the Dirichlet function $f(x) = \begin{cases} 1; & \text{if } x \in \{q_k : k \in \mathbb{N}\} \\ 0; & otherwise \end{cases}$ This function is not integrable on $[0,1]$.

EXAMPLE **11.** Let $\langle f_n \rangle$ be defined by $f_n(x) = \dfrac{x^{2n}}{1+x^{2n}}$, $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Therefore the limit function is given by

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^{2n}}{1+x^{2n}} = \begin{cases} 0; & |x| < 1 \\ \frac{1}{2}; & |x| = 1 \\ 1; & |x| > 1 \end{cases}$$

Note that each $f_n(x)$ is continuous on $\mathbb{R}$ but f is not continuous at ± 1.

Definition 2. [Pointwise Convergent Series of Functions] The series $\sum_{n=1}^{\infty} u_n(x)$ is said to be pointwise convergent (or simply convergent) on $E_0 \subset E$ with sum $s \in \mathcal{F}(E_0; \mathbb{R})^{\mathbb{N}}$ if the sequence $\langle s_n \rangle \in \mathcal{F}(E; \mathbb{R})^{\mathbb{N}}$ of partial sums converges (pointwise) to the function s on E_0. In other words, if

$$s(x) = \lim_{n\to\infty} s_n(x); \;\; \forall x \in E_0$$

EXAMPLE **12.** Consider, the series $\sum_{n=1}^{\infty} u_n(x)$, where, for each $n \in \mathbb{N}$, $u_n(x) = x^n$ for all $x \in (-1,1)$ and $u_0 = 1$. Then the series $\sum_{n=1}^{\infty} u_n(x)$ is (pointwise) convergent on $(-1,1)$ with sum

$$s(x) = \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$$

EXAMPLE **13.** Consider the series $\dfrac{x}{x+1} + \dfrac{x}{(x+1)(2x+1)} + \cdots ; x \geq 0$. Here

$$u_n(x) = \frac{1}{(n-1)x+1} - \frac{1}{nx+1}; \;\; S_n(x) = 1 - \frac{1}{nx+1}$$

Thus, when $x > 0$, $\lim_{n\to\infty} S_n(x) = 1$ and when $x = 0$, $\lim_{n\to\infty} S_n(x) = 0$ as $S_n(0) = 0$. Let $y = S_n(x)$, then $(y-1)(x+\frac{1}{n}) = -\frac{1}{n}$. The curve $y = s(x)$, when $x \geq 0$,

As $S_n(x)$ is certainly continuous, when the terms of the series are continuous, the approximation curves will always differ very materially from the curve $y = s(x)$, when the sum of the series is discontinuous.

consists of the part of the line $y = 1$ for which $x > 0$ and the origin. As n increases, this rectangular hyperbola (Fig. 5) approaches more and more closely to the lines $y = 1$, $x = 0$. If we reasoned from the shape of the approximate curves, we should expect to find that part of the axis of y for which $0 < y < 1$ appearing as a portion of the curve $y = s(x)$ when $x \geq 0$.

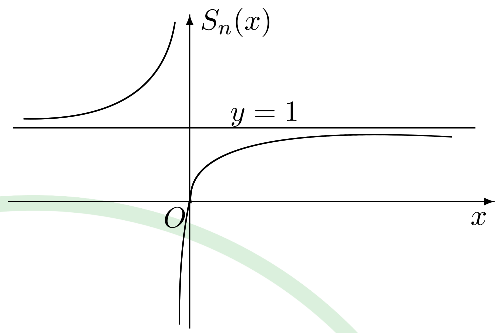

Figure 5: Example 13.

EXAMPLE **14.** Consider the series $\sum_{n=1}^{\infty} u_n(x)$, where, $u_n(x) = \dfrac{nx}{1+n^2x^2} - \dfrac{(n-1)x}{1+(n-1)^2x^2}$.

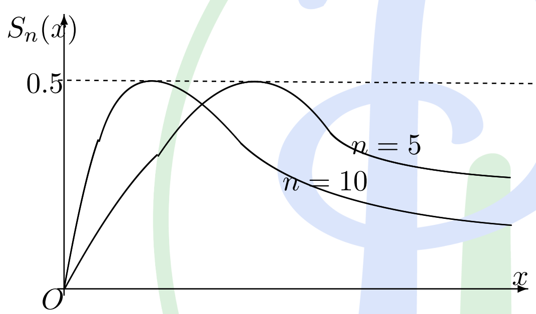

Figure 6: Example 14.

In this case, $S_n(x) = \dfrac{nx}{1+n^2x^2}$ and $\lim_{n\to\infty} S_n(x) = 0$ for all values of x. The sum function $s(x)$ of the series is continuous for all values of x, but we shall see that the approximation curves differ very materilly from the curve $y = s(x)$ in the neighbourhood of the origin. The curve $y = S_n(x)$ has a maximum at $(\frac{1}{n}, \frac{1}{2})$ and the minimum at $(-\frac{1}{n}, -\frac{1}{2})$ as depicted in the Fig. 6. The points on the axis of x just below the maximum and minimum move in towards the origin as n increases. And if we reasoned from the shape of the curves $y = S_n(x)$, we should expect to find the part of the axis of y from $-\frac{1}{2}$ to $\frac{1}{2}$ appearing as a portion of the curve $y = s(x)$.

EXAMPLE **15.** Find the sum function of $\sum_{n=1}^{\infty} (\cos x)^n$ on $(0, \pi)$.

Solution: Let $\langle S_n(x) \rangle$ be the sequence of partial sums of the series $\sum_{n=1}^{\infty} u_n(x)$, where, $u_n(x) = (\cos x)^n$. Thus

$$S_n(x) = \cos x + \cos^2 x + \cdots + \cos^n x = \frac{\cos x}{1-\cos x}\Big(1 - \cos^n x\Big)$$

$$\therefore \quad \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{\cos x}{1-\cos x}\Big(1-\cos^n x\Big) = \frac{\cos x}{1-\cos x}; \text{ as } |\cos x| < 1 \text{ for } 0 < x < \pi$$

Therefore, the sum function is given by $S(x) = \lim_{n\to\infty} S_n(x) = \frac{\cos x}{1-\cos x}$; for $0 < x < \pi$.

3 Uniform Convergence

The pointwise limit of a sequence of functions may differ radically from the functions in the sequence.

(i) In Examples 1 and 2, each f_n is continuous on $(-\infty, 1]$, but the limit function f is not continuous.

(ii) In Example 4, the graph of each f_n has two triangular spikes with heights that tend to ∞ as $n \to \infty$, while the graph of $f(x)$ (the x-axis) has none.

(iii) In Examples 5 and 6, each f_n is differentiable at x_0, while the limit function f is not differentiable at x_0 or even if $f'(x_0)$ exists, the $\langle f_n'(x_0)\rangle$ exists need not converge to $f'(x_0)$.

(iv) In Examples 8 and 9, each f_n is integrable, while the limit function f is non integrable in any compact interval.

There is nothing in Definitions 1 and 2 to preclude these apparent anomalies.

Definition 3. [Uniformly Convergent Sequence of functions]: Let $E \subset \mathbb{R}$. We say that a sequence $\langle f_n\rangle \in \mathcal{F}(E;\mathbb{R})^{\mathbb{N}}$ converges uniformly on $E_0 \subset E$ to a function $f : E_0 \to \mathbb{R}$ if, corresponding to an $\varepsilon > 0$, $\exists N = N(\varepsilon) \in \mathbb{N}$, depends on ε only, such that

$$|f_n(x) - f(x)| < \varepsilon; \quad \text{whenever } n \geq N \text{ and } \forall x \in E_0 \tag{3}$$

We interpret the uniform convergence in a geometric way. Draw the graphs of f_n and f.

Put a band of width ε around the graph of f. Condition (3) states that if ε is any positive number, then for $n > N$ the graph of $y = f_n(x)$ lies entirely below the graph of $f(x) + \varepsilon$ and entirely above the graph of $f(x) - \varepsilon$ as depicted on the Fig. 7. Thus draw a tube V of vertical radius ε around the graph f.

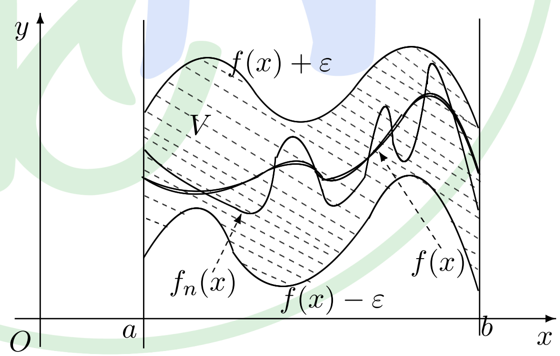

Figure 7: Geometrical significance of uniform convergence.

For n large, the graph of f_n is contained in the ε-tube V around the graph of f. Notice that the special feature of uniform convergence is that the rate at which $f_n(x)$ converges is independent of $x \in E$. For example,

EXAMPLE **16.** Consider, the sequence $\langle f_n(x)\rangle_n$ as in Example 1). The sequence $\langle f_n(x)\rangle_n$

Indeed, if for $\varepsilon = \frac{1}{10}$, the point $x_n = \sqrt[n]{\frac{1}{2}}$ is send by f_n to $\frac{1}{2}$ and thus not all points x satisfying Eq. nm (3), when n is large. The graph of f_n, as depicted in the figure 8, fails to lie in the ε-tube V. Here, $f_n(x)$ is converging very rapidly to zero for x near zero but arbitrarily slowly to zero for x near 1.

EXAMPLE **17.** Prove that for the sequence $\langle f_n\rangle$, $f_n \to f$ pointwise on a finite set $E(\subset \mathbb{R})$ the convergence is uniform.

of functions, where, $f_n : (0,1) \to \mathbb{R}$ is given by $f_n(x) = x^n$ for all $x \in [0,1]$, as depicted in the Figure 1. For each $x \in (0,1)$ it is clear that $f_n(x) \to 0$. The (pointwise) limit function, f, is

$$f(x) = \begin{cases} 0; & \text{if } 0 \le x < 1, \\ 1; & \text{if } x = 1 \end{cases}$$

Here the convergence is not uniform.

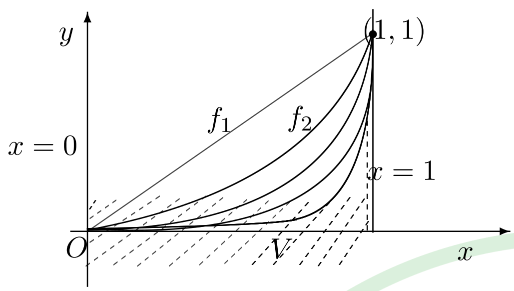

Figure 8: Non uniform, pointwise convergence.

Solution: Let $E = \{x_1, x_2, \cdots, x_n\}$ be a finite set in $\mathbb{R}$. Since the sequence $\langle f_n \rangle$ converges pointwise to f, so $\forall \varepsilon > 0$, $\exists N_k = N_k(x_k; \varepsilon)$ such that

$$|f_n(x_k) - f(x_k)| < \varepsilon; \forall n \ge N_k; k = 1(1)n.$$

Let for a pre-assigned $\varepsilon > 0$, $\max\limits_{x_k \in E} N_k(x_k; \varepsilon) = N(\varepsilon)$. Therefore,

$$|f_n(x) - f(x)| < \varepsilon; \forall n \ge N(\varepsilon) \text{ and } \forall x \in E$$

As $N(\varepsilon)$ depends on ε and not on x, $\langle f_n \rangle$ converges uniformly to f on E.

EXAMPLE **18.** Let $f_n(x) = \dfrac{x}{n}$ for $x \in \mathbb{R}$ as depicted in the Fig.. The sequence $\langle f_n(0) \rangle$ is a constant sequence $\langle 0 \rangle$. Hence the limit function is $f = 0$. More generally, if $a \in \mathbb{R}$, we get $\langle \frac{a}{n} \rangle$ as the pointwise sequence.

where, $f_n(x) = x^n$, for all $x \in [0;1]$ as depicted in the Fig. 9. For $x \in [0,1)$. We then have $\lim\limits_{n \longrightarrow \infty} f_n(x) = 0$. On the other hand, $\lim\limits_{n \longrightarrow \infty} f_n(1) = 1$. The sequence is therefore point wise convergent on $[0,1]$. We note, however, that the limit function

$$f(x) = \begin{cases} 0; & \text{if } 0 \le x < 1, \\ 1; & \text{if } x = 1 \end{cases}$$

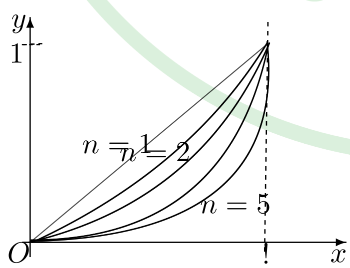

Figure 9: $f_n(x)$ for $n = 1, 2, 3, 4, 5$.

EXAMPLE **19.** *Show that* $\langle f_n'(x) \rangle$ *is uniformly convergent on* $[0,1]$, *where* $f_n(x) = \dfrac{\ln(1 + n^2x^2)}{n}$ *for* $x \in [0,1]$.

Solution: Consider the sequence of functions $\langle f_n \rangle$, defined by $f_n : [0,1] \to \mathbb{R}$, where, $f_n(x) = \dfrac{\log(1 + n^2x^2)}{n}; x \in [-k, k], k > 0$, as in the Fig. 10(a). Then $\langle f_n \rangle$ converges uniformly to $f(x) = 0$ on $[0,1]$. Now, $f_n'(x) = \dfrac{2nx}{1 + n^2x^2}$, for $x \in [0,1]$, as depicted in the Fig. 10(b). Clearly, $f_n(0) \to 0$, as $f_n(0) = 0$ for all $n \ge 1$. Let $x \ne 0$. For a given $\varepsilon > 0$, we see that

$$|f_n'(x) - 0| = \left| \frac{2nx}{1 + n^2x^2} \right| \le \frac{2}{n|x|} < \varepsilon$$

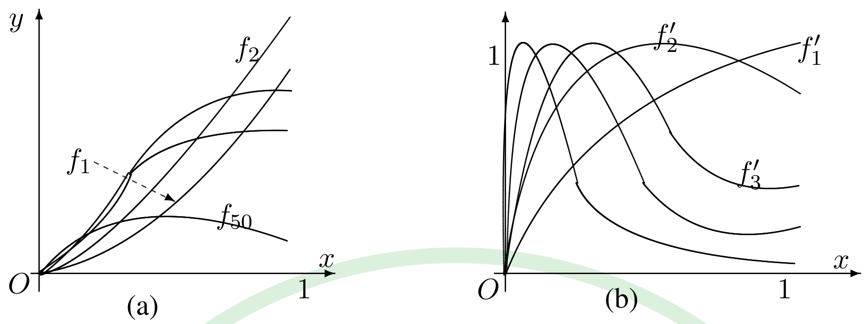

Figure 10: Figures of f_n and f_n'

whenever $n > \dfrac{2}{\varepsilon|x|}$. Let us take $N = [\dfrac{2}{\varepsilon|x|}] + 1 \in \mathbb{N}$, then

$$|f_n'(x) - 0| < \varepsilon \text{ for all } n \geq N$$

Therefore, $\langle f_n'(x) \rangle$ converges everywhere to the function $f'(x) = 0$ (Fig. 10(b)). Observe that N depends on both x and ε. Let

$$M_n = \sup_{x\in[0,1]} |f_n'(x) - f'(x)| = \sup_{x\in[0,1]} \frac{2x}{1+n^2x^2} = \sup_{x\in[0,1]} g(x).$$

Then $g'(x) = \frac{2(1-n^2x^2)}{(1+n^2x^2)^2}$, and $g'(x) = 0$ implies $x = \frac{1}{n} \in [0,1]$. Thus

$$M_n = \sup_{x\in[0,1]} |f_n'(x) - f'(x)| = \sup_{x\in[0,1]} \frac{2x}{1+n^2x^2} = f_n'(\frac{1}{n}) = \frac{1}{n}.$$

Hence, $M_n \to 0$ as $n \to \infty$, and consequently $\langle f_n' \rangle$ is uniformly convergent on $[0,1]$.

Definition 4. [Uniformly Convergent Series of Functions]: The series $\sum\limits_{n=1}^{\infty} u_n(x)$ is said to be uniformly convergent on E_0 if the sequence $\langle s_n \rangle$ of partial sums is uniformly convergent on E_0, i.e., corresponding to a $\epsilon > 0$, $\exists\ N(\epsilon) \in \mathbb{N}$ such that

$$\big|s_n(x) - s(x)\big| < \epsilon, \text{ whenever } n \geq N \text{ and } \forall x \in E_0 \tag{4}$$

where $s_n(x)$ is the n^{th} partial sum of the series $\sum\limits_{n=1}^{\infty} u_n(x)$.

THEOREM **1.** *If $\langle f_n \rangle$ converges uniformly to f on E, the $\langle f_n \rangle$ converges pointwise to f on E. The converse is not always true, i.e., pointwise convergence does not imply uniform convergence .*

Proof:

RESULT **1.** John Kelley refers to the growing steeple: Consider, the sequence $\langle f_n(x) \rangle_n$,

So there is no question of uniform convergence. Even if the function have compact domain of definition, and are uniformly bounded and uniformly continuous, pointwise convergence does not imply uniform convergence.

where, $f_n : [0,1] \to \mathbb{R}$ defined as, for $n \geq 1$,

$$f_n(x) = \begin{cases} n^2x; & \text{for } 0 \leq x \leq \frac{1}{n} \\ 2n - n^2x; & \text{for } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0; & \text{for } \frac{2}{n} \leq x \leq 1 \end{cases}$$

defined on $(-\infty, \infty)$, as depicted in the Fig. 11. Here $\lim_{n\to\infty} f_n(x) = 0$, for all x and f_n converges pointwise to the function $f = 0$. The graph of f_n, as depicted in the figure 11, fails to lie in the ε-tube V.

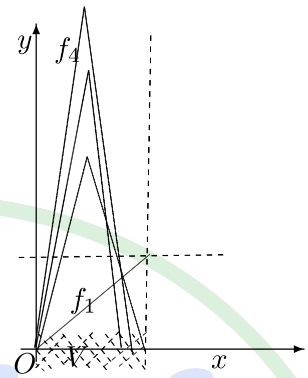

Figure 11: Graphical representation of $f_n(x)$

3.1 Test for Uniform Convergence

THEOREM **2** (Weiestrass M_n test)**.** *A sequence* $\langle f_n \rangle \in \mathcal{F}(E;\mathbb{R})^{\mathbb{N}}$*, where* $E \subset \mathbb{R}$*; converges uniformly on* $E_0 \subset E$ *if and only if, corresponding to an* $\varepsilon > 0$*,* $\exists N = N(\varepsilon) \in \mathbb{N}$*, depends on* ε *only, such that*

$$M_n = \sup\left\{|f_n(x) - f(x)| : x \in E_0\right\} < \varepsilon; \quad \textit{whenever } n \geq N$$

Proof: Necessary part : Let $\langle f_n(x)\rangle_n$ converges to $f(x)$ uniformly on E_0. Then corresponding to an $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\begin{aligned} & \left|f_n(x) - f(x)\right| < \epsilon, \text{ for } n \geq N, \ \forall x \in E_0 \\ \Rightarrow \quad & M_n = \sup\left\{|f_n(x) - f(x)| : x \in E_0\right\} < \varepsilon, \text{ for } n \geq N \\ \Rightarrow \quad & M_n \to 0 \text{ as } n \to \infty. \end{aligned}$$

Sufficient part : Let $M_n \to 0$ as $n \to \infty$, so corresponding to an $\epsilon > 0$, $\exists N_1 \in \mathbb{N}$ such that

$$\begin{aligned} & \left|M_n - 0\right| < \epsilon, \text{ for } n \geq N_1 \Rightarrow M_n < \epsilon, \text{ for } n \geq N_1 \\ \Rightarrow \quad & \sup\left\{|f_n(x) - f(x)| : x \in E_0\right\} < \varepsilon, \text{ for } n \geq N_1 \\ \Rightarrow \quad & \left|f_n(x) - f(x)\right| < \epsilon \text{ for } n \geq N_1; \forall x \in E_0 \end{aligned}$$

Therefore, $f_n(x) \to f(x)$ uniformly on E_0. □

EXAMPLE **20.** Consider, the sequence $\langle f_n(x)\rangle_n$, where, $f_n(x) = \dfrac{x}{1+nx^2}; x \in [a,b]$. For any $x \in [a,b]$,

$$f(x) = \lim_{n\longrightarrow\infty} f_n(x) = \lim_{n\longrightarrow\infty} \frac{x}{1+nx^2} = 0$$

Therefore, $\langle f_n(x)\rangle_n$ converges pointwise to zero on $[a,b]$. Now,

$$\left|f_n(x) - f(x)\right| = \left|\frac{x}{1+nx^2} - 0\right| = \frac{x}{1+nx^2} = g(x)(\text{say}).$$

For, $x > 0$, using A.M. $\geq$ G.M., we have

$$\frac{\frac{1}{x} + nx}{2} \geq \sqrt{\frac{1}{x} nx} \Rightarrow \frac{x}{1 + nx^2} \leq \frac{1}{2\sqrt{n}}$$

Therefore, $f_n(x) = \frac{1}{2\sqrt{n}}$, when $x = \frac{1}{\sqrt{n}} \in [a, b]$. Thus,

$$M_n = \sup\left\{\left|\frac{x}{1 + nx^2} - 0\right| : x \in [0, \infty)\right\} = \frac{1}{2\sqrt{n}} \to 0 \text{ as } n \to \infty$$

Therefore $\langle f_n(x) \rangle$ is uniformly convergent on $[a, b]$.

EXAMPLE **21.** Consider, the sequence $\langle f_n(x) \rangle_n$, where, $f_n(x) = x^n$ for all $x \in [0, 1]$. The (pointwise) limit function, f, is $f(x) = \begin{cases} 0; & \text{if } 0 \leq x < 1, \\ 1; & \text{if } x = 1 \end{cases}$ Now

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} \sup\left\{|x^n - f(x)| : x \in [0, 1]\right\} = \lim_{n\to\infty} 1 = 1 \neq 0$$

Therefore, the convergence is not uniform.

EXAMPLE **22.** Consider, the sequence $\langle f_n(x) \rangle_n$, where $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$, $n \in \mathbb{N}$, as depicted in the Figure 12, for all $x \in \mathbb{R}$. Here we clearly have $\lim_{n\longrightarrow\infty} f_n(x) = |x|$ for all $x \in \mathbb{R}$. So, the sequence $\langle f_n(x) \rangle_n$ has pointwise limit $f(x) = |x|$, for all $x \in \mathbb{R}$. Now

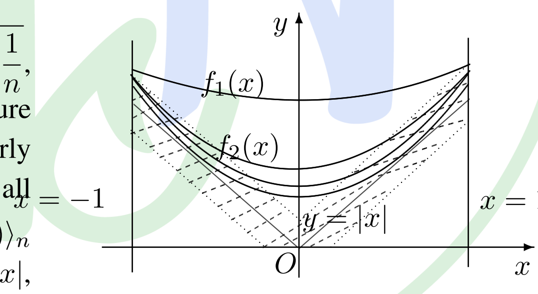

Figure 12: Graph of f_n as in example 22

$$M_n = \sup\left\{\left|\sqrt{x^2 + \frac{1}{n}} - |x|\right| : x \in \mathbb{R}\right\} = \sup\left\{\frac{1/n}{\sqrt{x^2 + \frac{1}{n}} + |x|} : x \in \mathbb{R}\right\} = \frac{1/n}{1/\sqrt{n}}$$

$$\therefore \quad \lim_{n\to\infty} M_n = \lim_{n\to\infty} \sup\left\{\left|\sqrt{x^2 + \frac{1}{n}} - |x|\right| : x \in \mathbb{R}\right\} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

Thus the convergence is uniform. We draw an ε-band around the limit function. Indeed, the graph of f_n, as depicted in the figure 12, lie in the ε-tube V. This also ensures geometrically that the convergence is uniform.

EXAMPLE **23.** The sequence $\langle f_n(x) \rangle_n$, where, $f_n(x) = \dfrac{\sin(n^2 x)}{n}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$ has pointwise limit $f(x) = 0$, for all $x \in \mathbb{R}$. Since $\left|\dfrac{\sin(n^2 x)}{n}\right| \leq \dfrac{1}{n}$, for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, so

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} \sup\left\{\left|\frac{\sin(n^2 x)}{n} - 0\right| : x \in \mathbb{R}\right\} = \lim_{n\to\infty} \frac{1}{n} = 0$$

Thus the convergence is uniform.

EXAMPLE **24.** Consider, the sequence $\langle f_n(x)\rangle_n$, where, $f_n(x) = \dfrac{x}{nx+1}$ for all $x \in [0,1]$. For a given $x \in [0,1]$,

$$f(x) = \lim_{n\longrightarrow\infty} f_n(x) = \lim_{n\longrightarrow\infty} \frac{x}{nx+1} = 0$$

Therefore $\langle f_n(x)\rangle$ converges pointwise to zero on $[0,1]$. Now,

$$\big|f_n(x) - f(x)\big| = \Big|\frac{x}{nx+1} - 0\Big| = \frac{x}{nx+1} = g(x)(\text{say}).$$

Then, $g'(x) = \frac{1}{(nx+1)^2} > 0, \forall x \in [0,1]$, so, $g(x)$ is strictly increasing function on $[0,1]$. Thus $g(x)$ attains its maximum value at $x = 1$. Therefore,

$$M_n = \sup_{x\in[0,1]} \big|f_n(x) - f(x)\big| = \sup_{x\in[0,1]} \frac{x}{nx+1} = \frac{1}{n+1}$$

Now , $M_n \to 0$ as $n \to \infty$. Therefore $\langle f_n(x)\rangle$ converges uniformly to 0 on $[0,1]$.

EXAMPLE **25.** Consider, the sequence of functions $\langle f_n(x)\rangle_n$, where, $f_n(x) = \dfrac{nx}{1+n^2x^2}$ for all

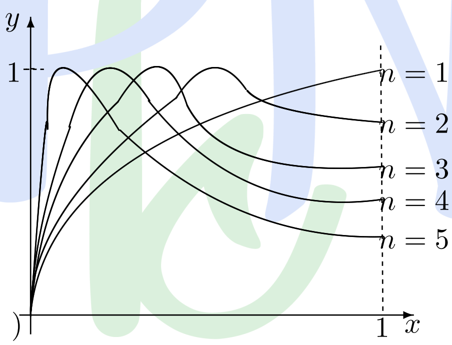

Figure 13: $f_n(x)$ for $n = 1,2,3,4,5$.

$x \in [a,b]$, containing 0. The graphs of f_n for $n = 1,2,3,4$ are shown in the Fig. 13. For any $x \in [a,b]$ containing zero,

$$f(x) = \lim_{n\longrightarrow\infty} f_n(x) = \lim_{n\longrightarrow\infty} \frac{nx}{1+n^2x^2} = 0$$

Therefore, $\langle f_n(x)\rangle_n$ is converges pointwise to zero on $[a,b]$. Now,

$$\big|f_n(x) - f(x)\big| = \Big|\frac{nx}{1+n^2x^2} - 0\Big| = \frac{nx}{1+n^2x^2} = g(x)(\text{say}).$$

Then, $g'(x) = \dfrac{n(1-n^2x^2)}{(1+n^2x^2)^2}$. Thus $g'(x) = 0 \Rightarrow x = \pm\frac{1}{n}$. Also

$$g''(x) = n\Big[\frac{(1+n^2x^2)^2\cdot(-2n^2x) + 2(1-n^2x^2)\cdot 2(1+n^2x^2)}{(1+n^2x^2)^4}\Big]$$

$$\text{or,} \qquad g''(x)\Big|_{x=\frac{1}{n}} = -2n^3x\cdot\frac{3-n^2x^2}{(1+n^2x^2)^4}\Big|_{x=\frac{1}{n}} = -\frac{n^2}{4} < 0$$

Thus $g(x)$ attains its maximum value at $x = 1/n$. Therefore,

$$M_n = \sup\Big\{\Big|\frac{nx}{1+n^2x^2} - 0\Big| : x \in [a,b]\Big\} = \frac{1}{2}$$

Now , $M_n \nrightarrow 0$ as $n \to \infty$. Therefore $\langle f_n(x) \rangle$ is not uniformly convergent on any interval containing 0. Also, The graph of f_n, as depicted in the figure 13, fails to lie in the ε-tube V, about $f = 0$, so that non uniform convergence is verified.

EXAMPLE **26.** Consider, the sequence $\langle f_n(x) \rangle_n$, where, $f_n(x) = f_n(x) = nxe^{-nx^2}, x \in [0, \infty)$. For any $x \in [0, \infty)$,

$$f(x) = \lim_{n \longrightarrow \infty} f_n(x) = \lim_{n \longrightarrow \infty} \frac{nx}{e^{nx^2}} = \lim_{n \longrightarrow \infty} \frac{x}{x^2 e^{nx^2}} = 0$$

Therefore, $\langle f_n(x) \rangle_n$ is converges pointwise to zero on $[0, \infty)$. Now,

$$\left| f_n(x) - f(x) \right| = \left| \frac{nx}{e^{nx^2}} - 0 \right| = \frac{nx}{e^{nx^2}} = g(x)(\text{say}).$$

Then, $g'(x) = \dfrac{n - 2n^2x^2}{e^{nx^2}}$. Thus $g'(x) = 0 \Rightarrow n - 2n^2x^2 = 0 \Rightarrow x = \pm \dfrac{1}{\sqrt{2n}}$. Also

$$g''(x)\Big|_{x=\frac{1}{\sqrt{2n}}} = \left[2n^2 x \frac{(2nx^2 - 3)}{e^{nx^2}} \right]_{x=\frac{1}{\sqrt{2n}}} < 0$$

Thus $g(x)$ attains its maximum value at $x = \frac{1}{\sqrt{2n}}$. Therefore,

$$M_n = \sup \left\{ \left| \frac{nx}{e^{nx^2}} - 0 \right| : x \in [0, \infty) \right\} = \sqrt{\frac{n}{2e}}$$

Now , $M_n \nrightarrow 0$ as $n \to \infty$. Therefore $\langle f_n(x) \rangle$ is not uniformly convergent on $[0, \infty)$.

EXAMPLE **27.** Thus for the sequence $\langle f_n \rangle$, $f_n \to f$ pointwise on a point set $E \subset \mathbb{R}$, the convergence is uniform.

EXAMPLE **28.** *Verify that the sequence* $\langle f_n \rangle$*, where* $f_n(x) = n \sin \sqrt{4\pi^2 n^2 + x^2}$ *converges uniformly on* $[0, k], k > 0$*. Does* $\langle f_n \rangle$ *converge uniformly on* $\mathbb{R}$*?*

Solution: The function $\sin \sqrt{4\pi^2 n^2 + x^2}$ can be written as

$$\begin{aligned} \sin \sqrt{4\pi^2 n^2 + x^2} &= \sin \left(2n\pi \sqrt{1 + \frac{x^2}{4\pi^2 n^2}} + 2n\pi - 2n\pi \right) \\ &= \sin 2n\pi \left(\sqrt{1 + \frac{x^2}{4\pi^2 n^2}} - 1 \right) \\ &= \sin \left(\sqrt{4\pi^2 n^2 + x^2} - 2n\pi \right) = \sin \frac{x^2}{\sqrt{4\pi^2 n^2 + x^2} + 2n\pi} \end{aligned}$$

Therefore, the limit function f is given by

$$\begin{aligned} f(x) &= \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} n \cdot \sin \frac{x^2}{\sqrt{4\pi^2 n^2 + x^2} + 2n\pi} \\ &= \lim_{n \to \infty} n \Big[\frac{x^2}{\sqrt{4\pi^2 n^2 + x^2} + 2n\pi} - \frac{1}{3!} \Big(\frac{x^2}{\sqrt{4\pi^2 n^2 + x^2} + 2n\pi} \Big)^3 + \cdots \Big] \\ &= \lim_{n \to \infty} \Big[\frac{x^2}{\sqrt{4\pi^2 + \frac{x^2}{n^2}} + 2\pi} - \frac{1}{3! n} \Big(\frac{x^2}{\sqrt{4\pi^2 + \frac{x^2}{n^2}} + 2\pi} \Big)^3 + \cdots \Big] = \frac{x^2}{4\pi} \end{aligned}$$

Now, using the fact that $\sin\alpha \geq \alpha - \dfrac{\alpha^3}{3!}$, we get for $x \in [0, k]$,

$$\begin{aligned}\left|\frac{x^2}{4\pi} - n\sin\sqrt{4\pi^2n^2 + x^2}\right| &= \left|\frac{x^2}{4\pi} - n\sin\frac{x^2}{\sqrt{4\pi^2n^2 + x^2} + 2n\pi}\right| \\ &\leq \left|\frac{x^2}{4\pi} - \frac{x^2}{\sqrt{4\pi^2n^2 + x^2} + 2n\pi} + \frac{1}{3!}\Big(\frac{x^2}{\sqrt{4\pi^2n^2 + x^2} + 2n\pi}\Big)^3\right| \\ &\leq \frac{k^2}{4\pi}\Big(1 - \frac{2}{1 + \sqrt{1 + \frac{k^2}{4\pi^2n^2}}}\Big) + \frac{n}{3!}\frac{k^5}{8n^3\pi^3} = M_n(\text{say})\end{aligned}$$

Now, $M_n \to 0$ as $n \to \infty$, so the given sequence of functions is uniformly convergent on $[0, k]$. For $x \in \mathbb{R}$, by the inequality $|\sin x| \leq |x|$, we obtain

$$\begin{aligned}\left|\frac{x^2}{4\pi} - n\sin\sqrt{4\pi^2n^2 + x^2}\right| &= \left|\frac{x^2}{4\pi} - n\sin\frac{x^2}{\sqrt{4\pi^2n^2 + x^2} + 2n\pi}\right| \\ &\geq \left|\frac{x^2}{4\pi} - \frac{x^2}{\sqrt{4\pi^2n^2 + x^2} + 2n\pi}\right| \geq \frac{x^2}{4\pi}\Big(1 - \frac{2}{1 + \sqrt{1 + \frac{x^2}{4\pi^2n^2}}}\Big),\end{aligned}$$

which shows that the convergence cannot be uniform on $\mathbb{R}$.

THEOREM **3** (Weierstrass M_n test for the series of functions)**.** *The series* $\sum_{n=1}^{\infty} u_n(x)$ *defined on* I *is uniformly convergent on* I *if* $\exists$ *a sequence of positive constants* $\{M_n\}$ *such that*

a) $\big|u_n(x)\big| \leq M_n, \forall n$

b) ΣM_n *is convergent*

Proof: Let S_n is the n^{th} partial sum of the series $\sum u_n(x)$. Then

$$S_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x) = \sum_{r=1}^{n} u_r(x).$$

Let $\epsilon > 0$ be chosen arbitrary,since $\sum M_n$ is convergent by Cauchy's criteria, $\exists N_0(\epsilon) \in \mathbb{N}$ such that

$$\big|P_n - P_m\big| < \epsilon, \text{ for } n > m > N_0,$$

where, $P_n = n^{th}$ partial sum of $\sum M_n$. Therefore,

$$M_{m+1} + M_{m+2} + \cdots + M_n < \epsilon, \text{ for } n > m > N_0$$

Therefore, if $n > m > N_0$ we obtain

$$\begin{aligned}\big|S_n(x) - S_m(x)\big| &= \Big|\sum_{r=1}^{n} u_r(x) - \sum_{r=1}^{m} u_r(x)\Big| = \Big|\sum_{r=m+1}^{n} u_r(x)\Big| \\ &\leq \sum_{r=m+1}^{n} \big|u_r(x)\big| \leq \sum_{r=m+1}^{n} M_r < \epsilon, \text{ for } n > m > N_0, \forall x \in I\end{aligned}$$

Therefore ,$\Sigma u_n(x)$ converges uniformly on I.

EXAMPLE **29.** Consider the series of functions $\sum_{n=1}^{\infty} \frac{\sin\ nx}{n^2}$ on $x \in (-\infty, \infty)$. Here

$$u_n(x) = \frac{\sin\ nx}{n^2} \Rightarrow \left|u_n(x)\right| \leq \frac{1}{n^2} = M_n, \forall x \in (-\infty, \infty)$$

Now, $\sum M_n = \sum \frac{1}{n^2}$ is a hyperhermonic series with $p = 2 > 1$, so convergent. Therefore, by Weierstrass M_n test, the given series of functions is uniformly convergent on $(-\infty, \infty)$.

EXAMPLE **30.** Consider the series of functions $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}; x \in [a, b]$.

$$u_n(x) = \frac{x}{n(1+nx^2)} \Rightarrow \left|\frac{x}{n(1+nx^2)}\right| < \frac{1}{n^2}$$

Now, $\sum M_n = \sum \frac{1}{n^2}$ is a hyper hermonic series with $p = 2 > 1$,so convergent. Then by M_n test the given series of functions $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ is uniformly convergent on $[a, b]$.

EXAMPLE **31.** Consider the series of functions $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}, p > 1, x \in \mathbb{R}$. Now

$$\left|\frac{\sin nx}{n^p}\right| \leq \frac{1}{n^p};\ \ \forall x \in \mathbb{R}$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is a Hyper-harmonic p-series with $p > 1$ and hence convergent. Therefore, by Weierstrass M-test the given series of functions is uniformly convergent for all $x \in \mathbb{R}$.

EXAMPLE **32.** Consider the series of functions $\sum_{n=1}^{\infty} \frac{\sin(nx^2 + x^2)}{n(n+1)}, x \in \mathbb{R}$. Now

$$\left|\frac{\sin(nx^2 + x^2)}{n(n+1)}\right| \leq \frac{1}{n(n+1)} < \frac{1}{n^2};\ \ \forall x \in \mathbb{R}$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a Hyper-harmonic p-series with $p = 2 > 1$ and hence convergent. Therefore, by Weierstrass M-test the given series of functions is uniformly convergent for all $x \in \mathbb{R}$.

EXAMPLE **33.** Test the covergence for $\Sigma \frac{x}{n^p + x^2 n^q}$ on any finite interval $[a, b]$.

Solution: First let $p > 1, q \geq 0$. Let $\alpha \geq \max\{|a|, |b|\}$, then

$$\left|u_n(x)\right| = \left|\frac{x}{n^p + x^2 n^q}\right| \leq \frac{\alpha}{n^p} = M_n \text{ (say)}$$

Now, $\sum M_n = \sum \frac{\alpha}{n^p}$ is a hyper harmonic series with $p > 2$, so convergent. By M_n test $\sum \frac{x}{n^p + x^2 n^q}$ is uniformly convergent on $[a, b]$.

Case -II: Let $0 < p \leq 1$ and $p + q \geq 2$. Then $\left|u_n(x)\right|$ attains its maximum value $\frac{1}{2n^{(p+q)/2}}$at the point where $x^2 n^q = n^p$. Now ,$\sum M_n$ is a hyper hermonic series converges as $p + q \geq 2$.

Hence by M_n test the given series uniformly convergent.

EXAMPLE **34.** Prove or disprove: $\sum_n 2^{-n}\cos(3^n x)$ represents an everywhere continuous function.

Solution: Let $s(x) = \sum_n 2^{-n}\cos(3^n x) = \sum_n u_n(x)$, where $u_n(x) = 2^{-n}\cos(3^n x); n \in \mathbb{N}$ and $x \in \mathbb{R}$. Thus

$$|u_n(x)| = |2^{-n}\cos(3^n x)| \le \frac{1}{2^n} = M_n, \text{ say}, \forall n \in \mathbb{N}$$

Now $\sum M_n = \sum \frac{1}{2^n} = 1$. So the series $\sum M_n$ is convergent. Hence, by Weierstrass M-test, the given series is uniformly convergent on $\mathbb{R}$. Again, $u_n(x) = 2^{-n}\cos(3^n x)$ is continuous $\forall x \in \mathbb{R}$ and $\forall n \in \mathbb{N}$. So the sum function f is everywhere continuous on $\mathbb{R}$.

EXAMPLE **35.** Consider the series $\sum u_n(x)$ for which the sum to first n-terms is $S_n(x) = \frac{\ln(1+n^4x^2)}{2n^2}; 0 \le x \le 1$. Here

$$S(x) = \lim_{n\to\infty} S_n(x) = 0;\ \ 0 \le x \le 1$$
$$\therefore S'(x) = 0;\ \ \forall x \in [0,1]$$

Again,

Thus we see that $\sum u_n'(x)$ does not converge uniformly on $[0,1]$ but the series may be differentiated term-by-term.

EXAMPLE **36.** Define a function $f(x) = \begin{cases} 0; & \text{if } x \le 0 \\ nx^2; & \text{if } 0 < x \le \frac{1}{2n} \\ x - \frac{1}{4n}; & \text{if } \frac{1}{2n} < x < \infty \end{cases}$ The sequence of functions $\langle f_n \rangle$ converge uniformly on the entire real line $\mathbb{R}$ to the function f, where, $f(x) = \begin{cases} 0; & \text{if } x \le 0 \\ x; & \text{if } x > 0 \end{cases}$ Notice that each of the functions f_n is continuously differentiable on the entire real line, but f is not differentiable at 0.

THEOREM **4** (Cauchy's criteria for Uniform Convergence)**.** *Let $E \subset \mathbb{R}$ and $\langle f_n \rangle \in \mathcal{F}(E;\mathbb{R})^{\mathbb{N}}$. Then $\langle f_n \rangle_n$ converges uniformly on $E_0 \subset E$ if and only if, corresponding to an $\varepsilon > 0$, $\exists N = N(\varepsilon) \in \mathbb{N}$, depends on ε only, such that*

$$\Big|f_m(x) - f_n(x)\Big| < \varepsilon;\quad \textit{whenever } m,n \ge N \textit{ and } \forall x \in E_0$$

Proof: Necessary part : Let $\langle f_n(x) \rangle_n$ converges uniformly on E_0 to a limit function $f(x)$. Then corresponding to an $\epsilon > 0$, $\exists$ positive integers $N_1, N_2 \in \mathbb{N}$ independent of x such that

$$\Big|f_n(x) - f(x)\Big| < \frac{\epsilon}{2}, \text{ for } n \ge N_1 \text{ and } \forall x \in E_0$$
$$\Big|f_m(x) - f(x)\Big| < \frac{\epsilon}{2}, \text{ for } m \ge N_2 \text{ and } \forall x \in E_0$$

Let $N = \max\{N_1, N_2\} \in \mathbb{N}$, then $\forall x \in E_0$ we have

$$\begin{aligned} \Big|f_m(x) - f_n(x)\Big| &= \Big|f_m(x) - f(x) + f(x) - f_n(x)\Big| \\ &\le \Big|f_m(x) - f(x)\Big| + \Big|f_n(x) - f(x)\Big| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \text{ for } n \ge N. \end{aligned}$$

Therefore, the condition holds.
Sufficient part : Conversely, if the condition of the theorem is satisfied, then, for each $x \in E_0$, the numerical sequence $\langle f_n(x)\rangle_n$ is a Cauchy sequence in $\mathbb{R}$ and hence converges. Let

$$f(x) = \lim_{n\to\infty} f_n(x); \;\; \forall x \in E_0.$$

For a given $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that

$$\Big|f_m(x) - f_n(x)\Big| < \varepsilon; \quad \text{whenever } m, n \geq N \text{ and } \forall x \in E_0$$

For fixed n, let $m \to \infty$ in the above equation, we find that

$$\Big|f(x) - f_n(x)\Big| < \varepsilon; \quad \text{whenever } n \geq N \text{ and } \forall x \in E_0$$

Since $\varepsilon > 0$ was arbitrary, it follows that, sequence of functions $\langle f_n(x)\rangle_n$ converges to f uniformly on E_0, as desired.

Deduction 3.1. A sequence $\langle f_n\rangle \in \mathcal{F}(E;\mathbb{R})^{\mathbb{N}}$, where $E \subset \mathbb{R}$; converges uniformly on $E_0 \subset E$ if and only if $\sup\Big\{|f_m(x) - f_n(x)| : x \in E_0\Big\} \to 0$ as $m, n \to \infty$.

EXAMPLE **37.** *Determine whether the sequence $\langle f_n\rangle$ of functions converges uniformly on E:*

a) $f_n(x) = \dfrac{x^2}{x^2 + (nx-1)^2}; E = [0,1]$ *b)* $f_n(x) = \sqrt{n+1}\sin^n x\;\cos x; E = \mathbb{R}$

c) $f_n(x) = \sqrt[n]{2^n + |x|^n}; E = \mathbb{R}$

Solution: a) The limit function f is given by

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2}{x^2 + (nx-1)^2} = 0.$$

Now, $f_n'(x) = \frac{2x(nx-1)^3}{[x^2+(nx-1)^2]^2}$, so, $f_n'(x) = 0$ gives $x = \frac{1}{n}$. Thus

$$M_n = \sup_{x\in[0,1]} \left|\frac{x^2}{x^2 + (nx-1)^2} - 0\right| = f_n\Big(\frac{1}{n}\Big) = 1.$$

As $M_n \not\to 0$ as $n \to \infty$, hence by Weierstrass M_n-test the given sequence of functions is not uniformly convergent on E.

b) The limit function f is given by

$$f(x) = \lim_{n\to\infty} f_n(x) = \cos x \lim_{n\to\infty} \sqrt{n+1}\sin^n x = 0.$$

Here, we use Weierstrass M_n-test. For that

$$M_n = \sup_{x\in\mathbb{R}} \Big|\sqrt{n+1}\sin^n x\;\cos x\Big| = \sqrt{n+1}\sup_{x\in\mathbb{R}} \Big|\sin^n x\;\cos x\Big|$$

Now we calculate supreme value of $g(x) = \sin^n x \ \cos x$, Now, $\log g(x) = n\log(\sin x) + \log(\cos x)$, thus the supreme value of g is given by the equation $\tan^2 x = n$. Thus,

$$\begin{aligned} M_n &= \sqrt{n+1}\sup_{x\in\mathbb{R}}\left|\sin^n x \ \cos x\right| = \sqrt{n+1}\cdot\left(\frac{n}{n+1}\right)^{n/2}\cdot\frac{1}{\sqrt{n+1}} \\ &= \left(\frac{n}{n+1}\right)^{n/2} \to \frac{1}{\sqrt{e}} \not\to 0 \text{ as } n\to\infty \end{aligned}$$

Hence by Weierstrass M_n-test the given sequence of functions is not uniformly convergent on E.

c) The limit function f is given by

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sqrt[n]{2^n + |x|^n} = \lim_{n\to\infty} 2\sqrt[n]{1+\left(\frac{|x|}{2}\right)^n} = \begin{cases} 2; & |x|\le 2 \\ |x|; & |x|>2 \end{cases}$$

We see that, all f_n's as well as the limit function is continuous, which implies that convergence is uniform.

EXAMPLE **38.** *Determine whether the sequence $\langle f_n\rangle$ of functions converges uniformly on E:*

a) $f_n(x) = \dfrac{1}{1+(nx-1)^2}; E = [0,1]$ *b)* $f_n(x) = nx^n(1-x); E = [0,1]$

c) $f_n(x) = \tan^{-1}\left(\dfrac{2x}{x^2+n^3}\right); E = \mathbb{R}$

Solution: a) The limit function f is given by

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{1}{1+(nx-1)^2} = \begin{cases} \frac{1}{2}; & x = 0 \\ 0; & x\in(0,1] \end{cases}$$

We see that, all f_n's are continuous on E, while the limit function is not continuous, which implies that convergence is not uniform.

b) Since $x^n(1-x)\to 0$ as $n\to\infty$, so, the limit function f is given by

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx^n(1-x) = 0.$$

Now, $f_n'(x) = nx^{n-1}[n-(n+1)x]$, so, $f_n'(x) = 0$ gives $x = \frac{n}{n+1}$. Thus

$$M_n = \sup_{x\in[0,1]}|nx^n(1-x) - 0| = f_n\left(\frac{n}{n+1}\right) = \frac{n^n}{(n+1)^{n+1}}$$

$$\therefore \quad \lim_{n\to\infty} M_n = \lim_{n\to\infty}\frac{1}{n}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^{n+1}} = \frac{1}{e}\cdot 0 = 0$$

Hence by Weierstrass M_n-test the given sequence of functions is uniformly convergent on E.

c) The limit function f is given by

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\tan^{-1}\left(\frac{2x}{x^2+n^3}\right) = 0$$

Now, $f_n'(x) = \frac{2n^3-2x^2}{4x^2+(x^2+n^3)^2}$, so, $f_n'(x) = 0$ gives $x = n\sqrt{n}$. Thus

$$M_n = \sup_{x\in\mathbb{R}}\left|\tan^{-1}\left(\frac{2x}{x^2+n^3}\right) - 0\right| = f_n\left(n\sqrt{n}\right) = \tan^{-1}\left(\frac{1}{n\sqrt{n}}\right)$$

Therefore, $M_n \to 0$ as $n\to\infty$, so by Weierstrass M_n-test the given sequence of functions is uniformly convergent on E to 0.

EXAMPLE **39.** *Determine whether the series* $\sum_{n=1}^{\infty} u_n(x)$ *of functions converges uniformly on* E*:*

a) $u_n(x) = \frac{\pi}{2} - \tan^{-1}\left(n^2(1+x^2)\right); E = \mathbb{R}$ *b)* $u_n(x) = 2^n \sin\frac{1}{3^n x}; E = (0, \infty)$

c) $u_n(x) = \ln\left(1+\frac{x^2}{n\cdot\ln^2 n}\right); E = (-k, k), k > 0$

Solution: a) We know, $\tan^{-1}x + \cot^{-1}x = \tan^{-1}x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}$. Using that identity, we get

$$\begin{aligned} u_n(x) &= \frac{\pi}{2} - \tan^{-1}\left(n^2(1+x^2)\right) = \tan^{-1}\frac{1}{n^2(1+x^2)} \\ &< \frac{1}{n^2(1+x^2)} \le \frac{1}{n^2} = M_n(\text{ say}); \forall x \in \mathbb{R}. \end{aligned}$$

Now, $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a hyperharmonic series with $p = 2(> 1)$, so convergent. Therefore, by Weierstrass M_n-test the given series of functions is uniformly convergent on $\mathbb{R}$.

b) For the problem, we use Cauchy criterion for uniform convergence. Let $S_n(x)$ be the n^{th} partial sum of the series, then it is given by

$$S_n(x) = 2\sin\frac{1}{3x} + 2^2\sin\frac{1}{3^2x} + \cdots + 2^n\sin\frac{1}{3^n x}$$

If $0 < \frac{1}{3^n x} \le \frac{\pi}{2}$, then

$$\begin{aligned} |S_{n+m}(x) - S_n(x)| &= 2^{n+1}\sin\frac{1}{3^{n+1}x} + \cdots + 2^{m+n}\sin\frac{1}{3^{m+n}x} \\ &\ge 2^{n+1}\frac{2}{\pi}\frac{1}{3^{n+1}x} + \cdots + 2^{m+n}\frac{2}{\pi}\frac{1}{3^{m+n}x} \\ &\ge 2^{n+1}\frac{2}{\pi}\frac{1}{3^{n+1}x} \end{aligned}$$

Putting, $x = \frac{1}{3^n}$, we obtain

$$\left|S_{m+n}\left(\frac{1}{3^n}\right) - S_n\left(\frac{1}{3^n}\right)\right| \ge \frac{2^{n+2}}{3\pi} \ge \frac{2^3}{3\pi}$$

Thus by Cauchy criteria, the series of functions does not converge uniformly on E.

c) Here, we use the Weierstrass M_n-test. Now

$$\begin{aligned} u_n(x) &= \ln\left(1+\frac{x^2}{n\cdot\ln^2 n}\right) = 1 + \frac{x^2}{n\cdot\ln^2 n} + \frac{x^4}{n^2\cdot\ln^4 n} + \cdots \\ &\le \frac{x^2}{n\cdot\ln^2 n} < \frac{k^2}{n\cdot\ln^2 n} = M_n \end{aligned}$$

By Cauchy's condensation test the series $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty}\frac{k^2}{n\cdot\ln^2 n}$ is convergent. Therefore, the given series of functions is uniformly convergent on E.

EXAMPLE **40.** *Determine whether the series* $\sum_{n=1}^{\infty} u_n(x)$ *of functions converges uniformly on* E*:*

a) $u_n(x) = \dfrac{x^n}{n!}; E = \mathbb{R}$ *b)* $u_n(x) = \dfrac{\sin(nx)}{\sqrt{n}}; E = [0, 2\pi]$

c) $u_n(x) = \dfrac{\cos^2(nx)}{n^2}; E = \mathbb{R}$

Solution: a) Let $S_n(x)$ be the n^{th} partial sum of the series. Then, for any $n \geq 1$, we have

$$\sup_{x\in\mathbb{R}} |S_n(x) - S_{n-1}(x)| = \sup_{x\in\mathbb{R}} |f_n(x)| \geq |f_n(n)| = \frac{n^n}{n!} \geq 1$$

Thus by Cauchy criteria, the series of functions does not converge uniformly on E.

b) First note that we do have pointwise convergence. Next notice that $\dfrac{2x}{\pi} \leq \sin x$ for any $x \in [0, \frac{\pi}{4}]$. Let $n \geq 10$ and $h \in \mathbb{N}$ such that $2n \leq n + h < n\sqrt{n}\frac{\pi}{4}$. Thus for any $k \in \mathbb{N}$ with $n \leq k < n + h$, we have $\dfrac{k}{n\sqrt{n}} < \dfrac{\pi}{4}$. Hence

$$\sin\left(\frac{k}{n\sqrt{n}}\right) \geq \frac{2}{\pi}\,\frac{k}{n\sqrt{n}} \Rightarrow \sum_{k=n}^{n+h} \frac{1}{\sqrt{k}} \sin\left(\frac{k}{n\sqrt{n}}\right) \geq \sum_{k=n}^{n+h} \frac{1}{\sqrt{k}}\frac{2}{\pi}\,\frac{k}{n\sqrt{n}}$$

$$\text{or,} \quad \sum_{k=n}^{n+h} \frac{1}{\sqrt{k}} \sin\left(\frac{k}{n\sqrt{n}}\right) \geq \sum_{k=n}^{n+h} \frac{2}{\pi}\,\frac{\sqrt{k}}{n\sqrt{n}} \geq \sum_{k=n}^{n+h} \frac{2}{\pi}\,\frac{\sqrt{n}}{n\sqrt{n}} \geq \frac{2}{\pi}$$

This obviously show that $\sup_{x\in[0,2\pi]} \left|\sum_{k=n}^{n+h} \frac{1}{\sqrt{k}} \sin\left(\frac{k}{n\sqrt{n}}\right)\right| \geq \frac{2}{\pi}$ for any $n \geq 0$ and $h \in \mathbb{N}$ such that $2n \leq n + h < n\sqrt{n}\frac{\pi}{4}$. Therefore, the convergence will not be uniform on $[0, 2\pi]$.

c) Here, we use the Weierstrass M_n-test. Now

$$\sup_{x\in\mathbb{R}} \Big|f_n(x) - 0\Big| = \sup_{x\in\mathbb{R}} \left|\frac{\cos^2(nx)}{n^2}\right| \leq \frac{1}{n^2} = M_n(\text{ say}); \forall x \in \mathbb{R}.$$

Now, $\sum\limits_{n=1}^{\infty} M_n = \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is a hyperharmonic series with $p = 2(> 1)$, so convergent. Therefore, by Weierstrass M_n-test the given series of functions is uniformly convergent on $\mathbb{R}$.

THEOREM **5.** *[Dini's Theorem of uniform convergence of a sequence] Let $I \subset \mathbb{R}$ be a compact interval and suppose that $\langle f_n\rangle \in \mathcal{F}(E;\mathbb{R})^{\mathbb{N}}$ is a sequence of continuous functions converging pointwise to a continuous function $f : I \to \mathbb{R}$. If $\langle f_n\rangle$ is increasing (i.e., $f_n(x) \leq f_{n+1}(x)$ for all $x \in I$ and $n \in \mathbb{N}$) or decreasing (i.e., $f_n(x) \geq f_{n+1}(x)$ for all $x \in I$ and $n \in \mathbb{N}$), then $\langle f_n\rangle$ converges to f uniformly on I.*

Proof: The uniform convergence of $\langle f_n\rangle$ to f is equivalent to the uniform convergence of $(f - f_n)$ (or $f_n - f$) to 0. Let $g_n = f - f_n$ (resp. $g_n = f_n - f$) if $\langle f_n\rangle$ is increasing (resp., decreasing). Then $\langle g_n\rangle$ is a decreasing sequence of continuous nonnegative functions converging pointwise to 0 on I. The theorem is proved if we show that this convergence is in fact uniform on I. Let $\varepsilon > 0$ be given. For each $x \in I$ $\lim g_n(x) = 0$ implies that we can pick $N(x) \in \mathbb{N}$ with $g_{N(x)}(x) < \dfrac{\varepsilon}{2}$. Since $g_{N(x)}(x)$ is continuous at x, there is a $\delta(x) > 0$ such that

$$g_{N(x)}(t) < \varepsilon; \quad \forall t \in (x - \delta(x), x + \delta(x)) \tag{5}$$

Since I is compact, we can cover I by a finite number of intervals $I_r = (x_r - \delta(x_r), x_r + \delta(x_r)); 1 \leq r \leq k$. Let $N = \max\{N(x_1), N(x_2), \cdots, N(x_k)\}$. Now, for any $t \in I$, we have $t \in I_r$ for some r and, by Eq. (5) $g_{N(x_r)}(t) < \frac{\varepsilon}{2}$. But since $N \geq N(x_r)$ and $\langle g_n \rangle$ is decreasing, we have

$$0 \leq g_N \leq g_{N(x_r)}(t) < \epsilon; \quad \forall t \in I$$

Therefore, we indeed have

$$g_n(x) \leq g_N(x) < \varepsilon; \quad \text{for } n \geq N \text{ and } \forall x \in I$$

and the proof is complete. □

EXAMPLE **41.** (i) Consider the sequence $\langle f_n \rangle$, where, $f_n(x) = x^{n-1}(1-x); x \in [0,1]$. Now

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^{n-1}(1-x) = 0; \text{ for } x \in [0,1]$$

The sequence $\langle f_n \rangle$ converges on $[0,1]$ to the function $f(x) = 0$ for $x \in [0,1]$. Each f_n is continuous on $[0,1]$, also $f(x)$ is continuous on $[0,1]$. Now, for each each $x \in [0,1]$

$$\begin{aligned} f_{n+1}(x) - f_n(x) &= (x^n - x^{n+1}) - (x^{n-1} - x^n) \\ &= -x^{n-1}(x-1)^2 \leq 0 \end{aligned}$$

i.e., $\langle f_n \rangle$ is monotone non-increasing for each $x \in [0,1]$. By Dini's Theorem 5, the convergence of the sequence is uniform on $[0,1]$.

(ii) Consider the sequence $\langle f_n \rangle$, where, $f_1(x) = \sqrt{x}$, $f_n(x) = \sqrt{x f_{n-1}(x)}$; for $n \geq 2$, $x \in [0,1]$. Therefore, $f_2(x) = \sqrt{x \cdot x^{1/2}} = x^{\frac{1}{2}+\frac{1}{2^2}}, \cdots, f_n(x) = x^{\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}}$.

At $x = 0$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^{1-\frac{1}{2^n}} = x$, i.e., $\langle f_n \rangle$ converges to f on $[0,1]$, where $f(x) = x$, $x \in [0,1]$. Each $f_(x)$ converges to f on $[0,1]$. Each $f_(x)$ is continuous on $[0,1]$ and the limit function is also continuous on $[0,1]$. Also

$$f_{n+1}(x) - f_n(x) = x^{\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}}\left[x^{\frac{1}{2^{n+1}}} - 1\right]$$

Therefore, $\langle f_n \rangle$ is monotone decreasing on $[0,1]$. By Dini's Theorem 5, the convergence of the sequence is uniform on $[0,1]$.

RESULT **2.** The following examples shows that each of the conditions (compactness of E, continuity of the limit function, continuity of f_n and monotonicity of the sequence $\langle f_n \rangle$) in Dini's theorem 5 is essential.

(i) To show that the compactness of E is essential, consider $f_n : (0,1) \to \mathbb{R}$ defined by $f_n(x) = \frac{1}{1+nx}$; for $x \in (0,1)$. The sequences has pointwise limit $f = 0$. Now

$$M_n = \sup_{x\in(0,1)} |f_n(x) - f(x)| = 1.$$

As $M_n \not\to 0$ as $n \to \infty$, therefore, the convergence is not uniform.

The assumption of continuity of f_n can not be omitted. Consider
$$f_n(x) = \begin{cases} 0; & \text{if } x = 0 \text{ or } \dfrac{1}{n} \le x \le 1 \\ 1; & \text{if } 0 < x < \dfrac{1}{n} \end{cases}$$
as depicted in the Fig. 14, are not continuous. They form a monotonic sequence pointwise convergent to zero on $[0, 1]$, but the convergence is not uniform.

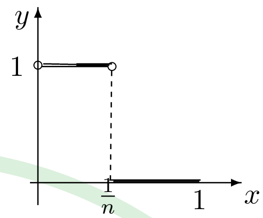

Figure 14: Graph of f_n

(ii) The continuity of the limit function is also essential. Indeed, the sequence $f_n : [0, 1] \to \mathbb{R}$, defined by $f_n(x) = x^n$; for $x \in [0, 1]$ fails to converge uniformly on $[0, 1]$ as in the Example 16.

(iii) Consider, $f_n : [0, 1] \to \mathbb{R}$ defined by

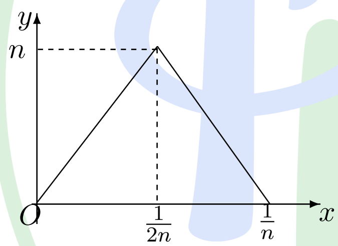

Figure 15: Graph of f_n

$$f_n(x) = \begin{cases} 2n^2x; & \text{if } 0 \le x \le \dfrac{1}{2n} \\ n - 2n^2\Big(x - \frac{1}{2n}\Big); & \text{if } \dfrac{1}{2n} < x \le \dfrac{1}{n} \\ 0; & \text{if } \dfrac{1}{n} < x \le 1 \end{cases}$$
as depicted in the Fig. 15

The functions $f_n(x)$ are continuous and form a sequence which is pointwise convergent to zero function on $[0, 1]$. Therefore, $\int_0^1 f(x)dx = 0$. Now, for each $n \in \mathbb{N}$,

$$\begin{aligned} \int_0^1 f_n(x)dx &= \int_0^{\frac{1}{2n}} 2n^2x\, dx + \int_{\frac{1}{2n}}^{1/n} \Big[n - 2n^2\Big(x - \frac{1}{2n}\Big)\Big]dx + \int_{1/n}^1 0\, dx \\ &= 2n^2\Big[\frac{x^2}{2}\Big]_0^{\frac{1}{2n}} + \Big[nx - 2n^2\Big(\frac{x^2}{2} - \frac{x}{2n}\Big)\Big]_{\frac{1}{2n}}^{1/n} = \frac{1}{2} \end{aligned}$$

Therefore, $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \frac{1}{2} \ne 0 \int_0^1 f(x)dx$. So the convergence is not uniform.

EXAMPLE **42.** *Let $u_n : [1, 2] \to \mathbb{R}$ be defined by $u_n(x) = \dfrac{x}{(1 + x)^n}$.*

(i) Show that $\sum_{n=1}^{\infty} u_n(x)$ converges for $x \in [1, 2]$

(ii) Use Dini's theorem to show that the convergence is uniform.

(iii) Does the following hold: $\int_1^2 \Big(\sum_{n=1}^{\infty} u_n(x)\Big)dx = \sum_{n=1}^{\infty} \int_1^2 u_n(x)dx$?

Solution: (a) Let $\frac{1}{|1+x|} < 1$, that is, $|1+x| > 1$. Then, we observe that

$$\sum_{n=1}^{\infty} \frac{x}{(1+x)^n} = x \sum_{n=1}^{\infty} \frac{1}{(1+x)^n} = \frac{x}{1+x} \cdot \frac{1}{1-\frac{1}{1+x}} = 1$$

So, in particular, $\sum\limits_{n=1}^{\infty} u_n(x)$ is convergent for $x \in [1,2]$.

(b) Let $E = [1,2]$, is compact and $u_n(x) \to 0$ pointwise. Clearly

$$u_{n+1} - u_n = \frac{x}{(1+x)^{n+1}} - \frac{x}{(1+x)^n} = -\frac{x^2}{(1+x)^{n+1}} < 0$$

so that the sequence is monotonic. All the hypotheses of Dini's Theorem 5 are satisfied and thus convergence is uniform.

(c) Since the convergence is uniform we can interchange the integral and summation. Thus the equality holds.

Two Important Theorems regarding the Test of Uniform Convergence

THEOREM **6** (Abel's test). *If*

(i) $b_n(x)$ *is positive monotone decreasing function of n for each fixed value* x *in* $[a,b]$

(ii) $\left|b_n(x)\right| < k, \forall x \in [a,b]$

(iii) The series $\Sigma u_n(x)$ *is uniformly convergent on* $[a,b]$ *then* $\Sigma b_n(x)u_n(x)$ *converges uniformly on* $[a,b]$

THEOREM **7** (Dirichlet's test). *The series* $\Sigma u_n(x)v_n(x)$ *will be uniformly convergent on a set* $E \subset \mathbb{R}$ *if*

(i) $\langle v_n \rangle_n$*is positive,a monotone decreasing sequence for every* $x \in E$ *and converges uniformly to zero on* E

(ii) $\left|S_n(x)\right| = \left|\sum\limits_{r=1}^{n} u_n(x)\right| < K$ *for every* $x \in E$ *and for* $\forall n \in \mathbb{N}$, *where* K *is a constant.*

Consider the following examples:

(i) Consider the series of functions $\sum\limits_{n=1}^{\infty} \frac{\sin nx}{n}$ defied on $[a,b]$, where $0 < a \le x \le b < 2\pi$. Let $u_n(x) = \sin nx$ and $v_n(x) = \frac{1}{n}$. Therefore

$$S_n(x) = \sum_{r=1}^{n} u_r(x) = \frac{\sin \frac{nx}{2}}{\sin \frac{x}{2}} \sin\left(x + \frac{n-1}{2}x\right)$$

$$\therefore \quad |S_n(x)| = |\sum_{r=1}^{n} u_r(x)| \le \left|\frac{1}{\sin \frac{x}{2}}\right| = \frac{1}{\sin \frac{x}{2}}; \quad 0 < \frac{x}{2} < \pi$$

Now, $\frac{1}{\sin \frac{x}{2}} = \operatorname{cosec}\frac{x}{2}$ is bounded for all $x \in [a,b]$, where $0 < a \le x \le b < 2\pi$. Also, $\langle v_n \rangle_n$is positive,a monotone decreasing sequence for every $x \in [a,b]$ and converges uniformly to zero for all $x \in [a,b]$.

Hence by Dirichlet's test the series converges uniformly on $[a,b]$.

4 Uniform Convergence and Limit Theorems

As was pointed out in the previous section, even if all functions in a sequence have a nice property (such as continuity, differentiability, etc.), the (pointwise) limit function, if it exists, need not (in general) share this property. Our goal now is to show that, if the convergence is uniform, then many nice properties satisfied by all the functions in the sequence will also be satisfied by their (uniform) limit.

4.1 Uniform Convergence and Limit

In general, limits do not commute. Since the integral is defined with a limit, and since we saw in the last section that integrals do not always respect limits of functions, we know some concrete instances of noncommutation of limits. The fact that continuity is defined with a limit, and that the limit of continuous functions need not be continuous, gives even more examples of situations in which limits do not commute.

THEOREM **8.** *Let $E_0 \subset E \subset \mathbb{R}$ and let $\langle f_n \rangle \in \mathcal{F}(E;\mathbb{R})^{\mathbb{N}}$. Suppose, $\langle f_n \rangle$ converges uniformly on E_0 to a function $f \in \mathcal{F}(E_0;\mathbb{R})$. Let $x_0 \in E_0$ and suppose that $\lim\limits_{x \longrightarrow x_0} f_n(x) = a_n$; $(n = 1, 2, \cdots)$ then*

(i) $\{a_n\}$ of real constants converges.

(ii) $\lim\limits_{x \longrightarrow x_0} f(x) = \lim\limits_{n \longrightarrow \infty} a_n$ i.e. $\lim\limits_{x \longrightarrow x_0} \{\lim\limits_{n \longrightarrow \infty} f_n(x)\} = \lim\limits_{n \longrightarrow \infty} \{\lim\limits_{x \longrightarrow x_0} f_n(x)\}$

Proof: (i) Let $\epsilon > 0$ be given. Since $\langle f_n \rangle$ converges uniformly on E_0 to a function $f \in \mathcal{F}(E_0;\mathbb{R})$, $\exists$ a positive number $N(\varepsilon) \in \mathbb{N}$ such that for all

$$\Big| f_m(x) - f_n(x) \Big| < \epsilon; \text{ whenever } m, n \geq N \text{ and } \forall x \in E_0$$

Keeping m, n fixed and let $x \longrightarrow x_0$ we get,

$$|a_m - a_n| < \epsilon\, ; n \geq N.$$

Hence, by Cauchy's general principle of convergence of real sequence of constants $\langle a_n \rangle$ converges, say to A, i.e., $\lim\limits_{n \longrightarrow \infty} a_n = A$. Therefore, $\langle a_n \rangle$ of real constants converges.

(ii) Let $\varepsilon > 0$ be chosen arbitrary. Since $\langle a_n \rangle$ converges to A $\exists$ a positive number $N = N(\varepsilon) \in \mathbb{N}$ such that

$$|a_n - A| < \frac{\epsilon}{3}\, ;\ \forall n \geq N$$

Since, $\langle f_n \rangle$ converges uniformly on E_0 to a function $f \in \mathcal{F}(E_0;\mathbb{R})$, $\exists$ a positive number $N = N(\varepsilon) \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}\, ;\ \forall n \geq N \text{ and } \forall x \in E_0$$

Again since $\lim\limits_{x \longrightarrow x_0} f_n(x) = a_n$ for all n, so corresponding to $\epsilon > 0$, $\exists \delta > 0$ such that

$$|f_n(x) - a_n| < \frac{\epsilon}{3};\ \text{whenever } 0 < |x - x_0| < \delta \text{ and } \forall n \in \mathbb{N}$$

Hence for all $n \geq N$ we have,

$$
\begin{aligned}
|f(x) - A| &\leq |f(x) - f_n(x)| + |f_n(x) - a_n| + |a_n - A| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon; \text{ whenever } 0 < |x - x_0| < \delta \\
\therefore \quad \lim_{x \longrightarrow x_0} f(x) &= A = \lim_{n \longrightarrow \infty} a_n.
\end{aligned}
$$

This proves the theorem. □

THEOREM **9.** *Let $E_0 \subset E \subset \mathbb{R}$ and let $\langle u_n \rangle \in \mathcal{F}(E;\mathbb{R})^{\mathbb{N}}$. Suppose, the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on E_0 to a sum function $s \in \mathcal{F}(E_0;\mathbb{R})$. Let $x_0 \in E_0$ and suppose that $\lim_{x \longrightarrow x_0} u_n(x) = a_n$; $(n = 1, 2, \cdots)$ then*

(i) $\sum_{n=1}^{\infty} a_n$ *converges and*

(ii) $\lim_{x \longrightarrow x_0} s(x) = \sum_{n=1}^{\infty} a_n$ *i.e.* $\lim_{x \longrightarrow x_0} \Big[\sum_{n=1}^{\infty} u_n(x)\Big] = \sum_{n=1}^{\infty} \Big[\lim_{x \longrightarrow x_0} u_n(x)\Big]$

Proof: (i) Since the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on E_0 corresponding to any $\epsilon > 0$, $\exists$ a positive integer m such that $\forall\, x \in [a, b]$ and for any integer $p \geq 1$,

$$\Big|\sum_{r=n+1}^{n+p} u_r(x)\Big| < \epsilon \,;\, \forall\, n \geq m \,,\; p \geq 1$$

Keeping n, p fixed, we let $x \longrightarrow x_0$ and obtain

$$\Big|\sum_{r=n+1}^{n+p} a_r\Big| < \epsilon \,;\, \forall\, n \geq m \,,\; p \geq 1$$

Hence it follows that the series $\sum_{n=1}^{\infty} a_n$ converges to a finite limit A (say).

(ii) Since $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly to $s(x)$ so corresponding to any $\epsilon > 0$, $\exists$ a positive integer m such that $\forall\, x \in [a, b]$ and for any integer $N_1 \in \mathbb{N}$, such that $\forall\, x \in [a, b]$,

$$\Big|\sum_{r=1}^{n} u_r(x) - s(x)\Big| < \frac{\epsilon}{3} \,;\, \forall\, n \geq N_1$$

Similarly, $\exists$ a positive integer $N_2 \in \mathbb{N}$ such that

$$\Big|\sum_{r=1}^{n} a_r - A\Big| < \frac{\epsilon}{3} \,;\, \forall\, n \geq N_2$$

Again, since $\lim\limits_{x\longrightarrow x_0} u_n(x) = a_n$; $(n = 1, 2, 3, \cdots)$, $\exists$ a suitable δ such that

$$\Big|u_n(x) - a_n\Big| < \frac{\epsilon}{3n} \text{ ; } \forall\, x \in |x - x_0| < \delta$$

$$\begin{aligned} \therefore \quad \Big|\sum_{r=1}^{n} u_r(x) - \sum_{r=1}^{n} a_r\Big| &\leq \sum_{r=1}^{n} \Big|u_r(x) - a_r\Big| \\ &< \frac{\epsilon}{3n} \cdot n = \frac{\epsilon}{3} \text{ ; } \forall\, x \in |x - x_0| < \delta \end{aligned}$$

Let $N = maxN_1, N_2 \in \mathbb{N}$ we get for $n \geq N$ and for $x \in |x - x_0| < \delta$, we have

$$\begin{aligned} \Big|s(x) - A\Big| &= \Big|s(x) - \sum_{r=1}^{n} u_r(x) + \sum_{r=1}^{n} u_r(x) - \sum_{r=1}^{n} a_r + \sum_{r=1}^{n} a_r - A\Big| \\ &\leq \Big|s(x) - \sum_{r=1}^{n} u_r(x)\Big| + \Big|\sum_{r=1}^{n} u_r(x) - \sum_{r=1}^{n} a_r\Big| + \Big|\sum_{r=1}^{n} a_r - A\Big| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

Therefore,

$$\lim_{x\longrightarrow x_0} f(x) = A = \sum_{n=1}^{\infty} a_n \; i.e. \; \lim_{x\longrightarrow x_0} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \Big\{\lim_{x\longrightarrow x_0} u_n(x)\Big\}$$

This proves the theorem. □

EXAMPLE **43.** For $n \geq 1$, define $f_n : [0, \frac{\pi}{2}] \to \mathbb{R}$ by $f_n(x) = \dfrac{4\cos^n x}{3 + \cos^n x}; x \in [0, \dfrac{\pi}{2}]$. Then each f_n is continuous. For $x \in (0, \frac{\pi}{2}]$, $\cos^n x \to 0$ as $n \to \infty$, and so the sequence $\langle f_n \rangle$ converges pointwise on $[0, \frac{\pi}{2}]$ to $f(x) = \begin{cases} 1; & \text{for } x = 0 \\ 0; & \text{for } x \in (0, \frac{\pi}{2}] \end{cases}$, which is not continuous on $[0, \frac{\pi}{2}]$. Since the limit function is not continuous, the sequence $\langle f_n \rangle$ cannot converge uniformly to $f(x)$ on $[0, \frac{\pi}{2}]$.

EXAMPLE **44.** Evaluate $\lim\limits_{x\to 0} \sum\limits_{n=1}^{\infty} \dfrac{\cos nx}{n(n+1)}$

Solution: The given series is of the form $\sum\limits_{n=1}^{\infty} u_n(x)$, where $u_n(x) = \dfrac{\cos nx}{n(n+1)}$. Now, for all $n \in \mathbb{N}$

$$\Big|u_n(x)\Big| = \Big|\frac{\cos nx}{n(n+1)}\Big| \leq \frac{1}{n(n+1)} < \frac{1}{n^2}; \;\; \forall x \in \mathbb{R}$$

Now, $\sum M_n = \sum \dfrac{1}{n^2}$ is a hyperhermonic series with $p = 2 > 1$, so convergent. Therefore, by Weierstrass M_n test, the given series of functions is uniformly convergent on $(-\infty, \infty)$. Thus

$$\begin{aligned} \lim_{x\to 0} \sum_{n=1}^{\infty} \frac{\cos nx}{n(n+1)} &= \sum_{n=1}^{\infty} \Big\{\lim_{x\to 0} \frac{\cos nx}{n(n+1)}\Big\} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \Big[\frac{1}{n} - \frac{1}{n+1}\Big] \\ &= \Big(1 - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) + \cdots + \Big(\frac{1}{n} - \frac{1}{n+1}\Big) + \cdots \\ &= \lim_{n\to\infty} \Big(1 - \frac{1}{n+1}\Big) = 1. \end{aligned}$$

THEOREM **10.** *Let a function $f_n : E \subseteq \mathbb{R} \to \mathbb{R}$ be bounded on $E_0 \subseteq E$ for all $n \in \mathbb{N}$. If the sequence $\langle f_n(x)\rangle$ of functions converges uniformly on E_0, then the limit function f is bounded on E_0 and the sequence $\langle f_n(x)\rangle$ is uniformly bounded on E_0.*

Proof: Since $\langle f_n(x)\rangle$ converges uniformly on E_0 to the limit function f, then for a preassigned $\varepsilon > 0$, there exists a natural number $N(\varepsilon) \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon, \text{ for all } n \geq N \text{ and } \forall x \in E_0$$

Therefore, $|f_M(x) - f(x)| < \varepsilon$. Since $f_M(x)$ is bounded on E_0, there exists a positive constant M such that $|f_N(x)| \leq K$. Thus, we have

$$\begin{aligned} |f(x)| &= |f_N(x) - \{f_N(x) - f(x)\}| \\ &\leq |f_N(x)| + |f_N(x) - f(x)| < M + \varepsilon; \text{ for all } x \in E_0 \end{aligned}$$

This proves that f is bounded on E_0. Now, for every $x \in E_0$ and each $n \geq n_0$

$$\begin{aligned} |f_n(x)| &= |f_n(x) - f(x) + f(x)| \\ &\leq |f_n(x) - f(x)| + |f(x)| < M + 2\varepsilon \end{aligned}$$

Again, f_n being bounded on E_0 for each $n \in \mathbb{N}$, we have $|f_n(x)| \leq M_n$ for every $x \in E_0$ and $n = 1, 2, \cdots, (n_0 - 1)$. Thus, if $M_0 = \min\{M_1, M_2, \cdots, M_{n_0-1}, M + 2\varepsilon\}$, then $|f_n(x)| \leq M_0$ for every $x \in E_0$ and for all $n \in \mathbb{N}$. Therefore, the sequence $\langle f_n(x)\rangle$ is uniformly bounded on E_0. $\square$

4.2 Uniform Convergence and Continuity

THEOREM **11.** *Let $E_0 \subset E \subset \mathbb{R}$ and let $\langle f_n\rangle \in \mathcal{F}(E;\mathbb{R})^{\mathbb{N}}$. If each f_n is continuous at some $x_0 \in E_0$ and $\langle f_n\rangle$ converges uniformly on E_0 to a function $f \in \mathcal{F}(E_0;\mathbb{R})$; then f is also continuous at x_0. Thus, if each f_n is continuous on E_0, then so is the limit function f.*

Proof: Let $\varepsilon > 0$ be chosen arbitrary. Since f is the uniform limit of $\langle f_n\rangle$, we can find $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\Big|f_n(x) - f(x)\Big| < \frac{\epsilon}{3}\,;\ \forall\, n \geq N, \forall\, x \in E_0 \qquad (i)$$

With N as in (i), the continuity of f_N at x_0 implies that we can find $\delta = \delta(\epsilon) > 0$ with

$$|f_N(x) - f_N(x_0)| < \frac{\epsilon}{3}\,;\ \forall\, x \in E_0 \cap (x_0 - \delta, x_0 + \delta) \qquad (ii)$$

Also, (i) implies that

$$\Big|f_N(x) - f(x)\Big| < \frac{\epsilon}{3}\,;\ \forall\, x \in E_0 \cap (x_0 - \delta, x_0 + \delta) \qquad (iii)$$

Now (i), (ii) and (iii) imply that, for each $x \in E_0 \cap (x_0 - \delta, x_0 + \delta)$, we have

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_N(x) + f_N(x) - f_N(x_0) + f_N(x_0) - f(x_0)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \varepsilon \end{aligned}$$

and hence f is continuous at x_0. Since x_0 is taken arbitrary on E_0, so the result holds. $\square$

THEOREM **12.** *Let $E_0 \subset E \subset \mathbb{R}$ and let $\langle u_n \rangle \in \mathcal{F}(E;\mathbb{R})^{\mathbb{N}}$. If each f_n is continuous at some $x_0 \in E_0$ and the series $\sum\limits_{n=1}^{\infty} f_n$ converges uniformly on E_0 to a sum function $s \in \mathcal{F}(E_0;\mathbb{R})$; then s is also continuous at x_0. In particular, if each f_n is continuous on E_0, then so is the limit function s.*

Proof: Let $\varepsilon > 0$ be chosen arbitrary. x_0 be arbitrary point on E_0. Since $\sum u_n(x)$ converges uniformly to $s(x)$ on E_0, therefore for $\epsilon > 0$ we can chose $N \in \mathbb{N}$ such that $\forall\, x \in E_0$

$$\Big|\sum_{r=1}^{n} u_r(x) - s(x)\Big| < \frac{\epsilon}{3}\ ;\ \forall\, n \geq N$$

and is particular, at $x = x_0 \in E_0$ and $n = N$,

$$\Big|\sum_{r=1}^{N} u_r(x_0) - s(x_0)\Big| < \frac{\epsilon}{3}$$

Again, since each $u_n(x)$ is continuous at x_0, the sum of a finite number of functions $\sum\limits_{r=1}^{n} u_r(x)$ is also continuous at $x = x_0$. Therefore, for $\epsilon > 0$, $\exists \delta > 0$ such that

$$\Big|\sum_{r=1}^{N} u_r(x) - \sum_{r=1}^{N} u_r(x_0)\Big| < \frac{\epsilon}{3}\ ;\ \forall\, x \in E_0 \cap (x_0 - \delta, x_0 + \delta)$$

Hence for $\forall\, x \in E_0 \cap (x_0 - \delta, x_0 + \delta)$, we have

$$\begin{aligned}
\Big|s(x) - s(x_0)\Big| &= \Big|s(x) - \sum_{r=1}^{N} u_r(x) + \sum_{r=1}^{N} u_r(x) - \sum_{r=1}^{N} u_r(x_0) + \sum_{r=1}^{N} u_r(x_0) - u(x_0)\Big| \\
&\leq \Big|s(x) - \sum_{r=1}^{N} u_r(x)\Big| + \Big|\sum_{r=1}^{N} u_r(x) - \sum_{r=1}^{N} u_r(x_0)\Big| + \Big|\sum_{r=1}^{N} u_r(x_0) - u(x_0)\Big| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

$\therefore s(x) \longrightarrow s(x_0)$ as $x \longrightarrow x_0$. Since x_0 is arbitrary, so $s(x)$ is continuous in E_0. □

RESULT **3.** The converse of this theorem is not always true as may be seen in the following example

(i) The series $\sum\limits_{n=1}^{\infty} \dfrac{x}{(nx+1)\{(n-1)x+1\}}$ is uniformly convergent on any finite interval $[a, b]$, when $0 < a < b$. But the series is only point wise convergent but not uniformly convergent on $[a, b]$. Here $u_n(x) = \dfrac{x}{(nx+1)\{(n-1)x+1\}} = \dfrac{1}{(n-1)x+1} - \dfrac{1}{nx+1}$. Therefore

$$\begin{aligned} s_n(x) &= u_1(x) + u_2(x) + u_3(x) + \cdots + u_n(x) \\ &= \left(1 - \frac{1}{x+1}\right) + \left(\frac{1}{x+1} - \frac{1}{2x+1}\right) + \left(\frac{1}{2x+1} - \frac{1}{3x+1}\right) + \\ &\qquad \cdots + \left[\frac{1}{(n-1)x+1} - \frac{1}{nx+1}\right] \\ &= 1 - \frac{1}{nx+1} \\ \therefore s(x) &= \lim_{n\longrightarrow\infty} s_n(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases} \end{aligned}$$

$\therefore$ The sum function $s(x)$ is discontinuous on $[a, b]$, and therefore the converges is uniform on $[a, b]$, it is only point wise. When $x \neq 0$, let $\epsilon > 0$ be given

$$\begin{aligned} |s_n(x) - s(x)| &= \left|1 - \frac{1}{nx+1} - 1\right| \\ &= \left|\frac{1}{nx+1}\right| < \epsilon\,; \text{ whenever } n > \frac{1}{x}\left(\frac{1}{\epsilon} - 1\right) \end{aligned}$$

But $\frac{1}{x}\left(\frac{1}{\epsilon} - 1\right)$ decreases with x. Hence if we take $N = \left[\max_{x\in[a,b]}\left\{\frac{1}{x}\left(\frac{1}{\epsilon} - 1\right)\right\}\right] + 1 \in \mathbb{N}$, which is independent of x, then we obtain $|s_n(x) - s(x)| < \epsilon$, whenever $n > N$ for all $x \in [a, b]$ i.e. the series converges uniformly to $s(x) = 1$ on $[a, b]$.

Below are some examples:

(i) Consider the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^4}{(1+x^4)^n} = x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \frac{x^4}{(1+x^4)^3} + \cdots$ on $[0, 1]$. Let $\{s_n(x)\}_n$ be the sequence of n^{th} partial sums. Then

$$\begin{aligned} s_n(x) &= x^4\left\{1 + \frac{1}{1+x^4} + \frac{1}{(1+x^4)^2} + \cdots\right\} \\ &= x^4 \frac{1 - \frac{1}{(1+x^4)^n}}{1 - \frac{1}{1+x^4}} = (1+x^4)\left\{1 - \frac{1}{(1+x^4)^n}\right\}, x \neq 0. \end{aligned}$$

Therefore $\lim_{n\to\infty} s_n(x) = 1 + x^4$ i.e., $s(x) = 1 + x^4, x \neq 0$. Again, at $x = 0, s_n(0) = 0$. $s(0) = \lim_{n\to\infty} s_n(0) = 0$. So

$$s(x) = \begin{cases} 0; & x = 0 \\ 1 + x^4; & x \neq 0 \end{cases}$$

$\lim_{n\to\infty} s(x) = 1 \neq s(0) \Rightarrow s$ is not continuous at $x = 0$ i.e., on $[0, 1]$, and therefore the converges is not uniform on $[0, 1]$, it is only point wise. Given series is a series of continuous functions on $[0, 1]$ but its sum function s is not so. Hence, the series does not converge uniformly on $[0, 1]$.

(ii) Consider the series $\sum_{n=0}^{\infty} \frac{x^4}{(1+x^4)^n}$ on $[0,1]$. Here,

$$s(x) = \begin{cases} 1; & x \neq 1 \\ 0; & x = 0 \end{cases}$$

Thus the sum function $s(x)$ is discontinuous on $[0,1]$, and therefore the converges is not uniform on $[0,1]$, it is only point wise.

EXAMPLE **45.** A function S defined by $S(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{5^n}; x \in \mathbb{R}$. Show that S is continuous for any $x \in \mathbb{R}$.

Solution: The given series is of the form $\sum u_n(x)$, where, $u_n(x) = \frac{\cos nx}{5^n}$. Now

$$|u_n(x)| = \left|\frac{\cos nx}{5^n}\right| \leq \frac{1}{5^n} = M_n, \text{ say}$$

$$\sum M_n = 1 + \frac{1}{5} + \frac{1}{5^2} + \cdots = \frac{1}{1-\frac{1}{5}} = \frac{5}{4}$$

So the series $\sum M_n$ is convergent. Hence, by Weierstrass M-test the series $\sum_{n=1}^{\infty} \frac{\cos nx}{5^n}$ is uniformly convergent on $\mathbb{R}$. Again $\frac{\cos nx}{5^n}$ is continuous for all $x \in \mathbb{R}$ and $\forall n \in \mathbb{N}$. So that the sum function $S(x)$ is continuous on $\mathbb{R}$.

4.3 Uniform Convergence and Integration

We shall investigate when do we have $\lim_{n\longrightarrow\infty} \int_a^b f_n(x)dx = \int_a^b \left\{\lim_{n\longrightarrow\infty} f_n(x)\right\}dx = \int_a^b s(x)dx.$ Consider the sequence $\langle f_n(x)\rangle$ given by, $f_n(x) = 2nxe^{-nx^2}$.Each $f_n(x)$ is continuous on $[0,1]$ and hence integrable there. Now,

$$f(x) = \lim_{n\longrightarrow\infty} f_n(x) = \lim_{n\longrightarrow\infty} \frac{2nx}{e^{nx^2}} = 0\,;\ \forall\, x \in [0,1]$$

But

$$\int_0^1 f_n(x)dx = -\int_0^1 e^{-nx^2}(-2nx)dx$$

$$= -\int_0^1 e^{-nx^2} d(-nx^2) = -\left[e^{-nx^2}\right]_0^1 = 1 - e^{-n}$$

$$\therefore \quad \lim_{n\longrightarrow\infty} \int_0^1 f_n(x)dx = \lim_{n\longrightarrow\infty} (1-e^{-n}) = 1$$

Also,

$$\int_0^1 \left\{\lim_{n\longrightarrow\infty} f_n(x)\right\}dx = \int_0^1 0\cdot dx = 0$$

$$\therefore \quad \lim_{n\longrightarrow\infty} \int_0^1 f_n(x)dx \neq \int_0^1 \left\{\lim_{n\longrightarrow\infty} f_n(x)\right\}dx.$$

We show that Riemann integrability is preserved when we pass to uniform limits.

RESULT **4.** [Lebesgue's Integrability Criterion:] Let $f : [a, b] \to R$ be a bounded function. Then f is Riemann integrable if and only if it is continuous almost everywhere.

Proof: For each $N \in \mathbb{N}$, let $D_n = \{x \in [a, b] : \omega_f(x) \geq \frac{1}{N}\}$ and put $D = \bigcup_{n=1}^{\infty} D_n$. Then D is the set of all discontinuity points of f in $[a, b]$(?). Suppose that $f \in \mathcal{R}[a, b]$. We want to prove that each D_n has measure zero. By Riemann's Lemma, given any $\varepsilon > 0$ we can find a partition $P = \left(x_k\right)_{k=0}^{n}$ of $[a, b]$ such that $U(P; f) - L(P; f) < \frac{\varepsilon}{N}$. Let us divide $\{1, 2, \cdots, n\}$ into two parts, $G_i = G_i(D_N); i = 1, 2$

$$G_1 = \{j : (x_{j-1}, x_j) \cap D_N \neq \phi\};\;\; G_2 = \{j : (x_{j-1}, x_j) \cap D_N = \phi\}$$

Now, with $M_r = \sup\{f(x) : x \in [x_{r-1}, x_r]\}$, $m_j = \inf\{f(x) : x \in [x_{r-1}, x_r]\}$ and $\delta_r = x_r - x_{r-1}$, we have

$$U(P; f) - L(P; f) = \sum_{r \in G_1} (M_r - m_r)\delta_r + \sum_{r \in G_2} (M_r - m_r)\delta_r < \frac{\varepsilon}{N}$$

Since $(x_{r-1}, x_r) \cap D_N \neq \phi$, implies that $M_r - m_r \geq \dfrac{\varepsilon}{N}$, we have

$$\sum_{r \in G_1} \delta_r \leq N \sum_{r \in G_1} (M_r - m_r)\delta_r < \frac{N\varepsilon}{N} = \varepsilon$$

But the intervals (x_{r-1}, x_r) with $r \in G_1$ cover D_N. Therefore, D_N has measure zero for each N; and hence, D has measure zero.

Conversely, let us assume that D has measure zero and let $\varepsilon > 0$ be given. Each $[a, b]/D_N$ is (relatively) open. Therefore each D_N is a closed (hence compact) subset of $[a, b]$ and has measure zero. Let N be such that $(b - a)/ = N < \varepsilon/2$ and pick a partition $P = \left(x_k\right)_{k=0}^{n}$ of $[a, b]$ such that $\sum_{r \in G_1} \delta_r < \dfrac{\varepsilon}{4M}$, where, $M = \sup\{|f(x)| : x \in [a, b]\}$. Next, note that if $K = \bigcup_{r \in G_2} [x_{r-1}, x_r]$, with G_2 defined as above, then K is a compact subset of $[a, b]$ such that $x \in K$ implies $\omega_f(x) < \frac{1}{N}$. Thus, we can pick a $\delta > 0$, such that

$$|s - t| < \delta \Rightarrow |f(s) - f(t)| < \frac{1}{N}.$$

Let $Q = \left(x'_{k'}\right)_{k'=0}^{n'}$ be a refinement of P with mesh $\nu(P) < \delta$. Then, with $M_{r'}, m_{r'}$ and $\delta'_{r'}$, defined as usual and the subsets $G'_1, G'_2 \subset \{1, 2, \cdots n'\}$ defined as in the first part of the proof, we have

$$\begin{aligned} U(Q; f) - L(Q; f) &= \sum_{r' \in G'_1} (M'_r - m'_r)\delta'_r + \sum_{r' \in G'_2} (M'_r - m'_r)\delta'_r \\ &< 2M \sum_{r \in G_1} \delta_r + \frac{b - a}{N} < 2M \cdot \frac{\varepsilon}{4M} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

which shows indeed that $f \in \mathcal{R} \in [a, b]$ and completes the proof. □

THEOREM **13** (Uniform Convergence and Integrability)**.** *Let $\langle f_n \rangle$ be a sequence of Riemann integrable functions on a compact interval $[a,b] \subset \mathbb{R}$. If $\lim f_n = f$, uniformly on $[a,b]$, then f is also Riemann integrable on $[a,b]$ and we have*

$$\int_a^x f(t)\,dt = \lim_{n\to\infty} \int_a^x f_n(t)\,dt;\ \ \forall x \in [a,b]$$

Proof: Let $\varepsilon > 0$ be chosen arbitrary. The uniform convergence of $\langle f_n \rangle$ to f implies that, for some $n \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon;\ \ \forall x \in [a,b] \text{ and } n \geq N$$

In particular, $|f_N(x) - f(x)| < \varepsilon$, for all $x \in [a,b]$. Therefore,

$$\begin{aligned} |f(x)| &= |f(x) - f_N(x) + f_N(x)| \leq |f(x) - f_N(x)| + |f_N(x)| \\ &< \varepsilon + |f_N(x)|;\ \ \forall x \in [a,b] \end{aligned}$$

Now, for each $n \in \mathbb{N}$, f_n is Riemann integrable and hence continuous on $[a,b]$ except on a set D_n of measure zero. Let $D = \bigcup_{n=1}^{\infty} D_n$. Then D has measure zero. For each $x \in [a,b]/D$, all the f_n are continuous at x. Since f_n converges to f uniformly, Theorem implies that f is also continuous at x. Thus, f is indeed continuous on $[a,b]/D$ and hence Riemann integrable.

Next, given any $\varepsilon > 0$, by uniform convergence, we can find $N \in \mathbb{N}$ such that $|f_N(t) - f(t)| < \frac{\varepsilon}{b-a}$, for all $[a,b]$.

$$\Big|\int_a^x f(t)\,dt - \int_a^x f_N(t)\,dt\Big| \leq \int_a^b |f(t) - f_N(t)|\,dt < \varepsilon$$

and the proof is complete. □

THEOREM **14.** *Let $\langle f_n(x) \rangle$ be a sequence of R-integrable functions on $[a,b]$ where a,b are finite. If $\langle f_n(x) \rangle$ converges uniformly to $f(x)$ which is R-integrable in $[a,b]$, then*

$$\lim_{n\longrightarrow\infty} \int_a^b f_n(x)dx = \int_a^b \Big\{\lim_{n\longrightarrow\infty} f_n(x)\Big\}dx = \int_a^b f(x)dx$$

Proof: Let $\epsilon > 0$ be any given positive number. Then by definition of U.C of the $\{s_n(x)\}$ on $[a,b]$ We can find a positive integer $N(\epsilon)$ for which $\Big|s_n(x) - s(x)\Big| < \frac{\epsilon}{b-a}$; $\forall\, n > N$ and for all $x \in [a,b]$.
We chose $n > N$, we have

$$\begin{aligned} \Big|\int_a^b s_n(x)dx - \int_a^b s(x)dx\Big| &= \Big|\int_a^b \{s_n(x) - s(x)\}\Big| \\ &\leq \int_a^b \Big|s_n(x) - s(x)\Big|dx \\ &\leq \int_a^b \frac{\epsilon}{b-a}dx = \epsilon \end{aligned}$$

$\therefore \lim\limits_{n\longrightarrow\infty} \int_a^b s_n(x)dx = \int_a^b \Big\{\lim\limits_{n\longrightarrow\infty} s_n(x)\Big\}dx = \int_a^b s(x)dx$. Hence the theorem.

THEOREM **15** (Term by Term Integration of an Uniform Convergent Series)**.** *Let $\langle u_n(x)\rangle$ be a sequence of R-integrable functions on a compact interval $[a,b] \subset \mathbb{R}$. If the infinite series $\sum_{n=1}^{\infty} u_n$ converges uniformly to sum $s(x)$ on $[a,b]$, then*

(i) $s \in \mathcal{R}[a,b]$, i.e., s is R-integrable on $[a,b]$, and

(ii) $\int_a^b s(x)dx = \int_a^b \Big[\sum_{n=1}^{\infty} u_n(x)\Big]dx = \sum_{n=1}^{\infty}\Big[\int_a^b u_n(x)dx\Big]$

Proof: Here $s_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x) =$ the n^{th} partial sum of the series. Let $\epsilon > 0$ be any given positive number. Then by definition of uniformly convergent of the $\langle s_n(x)\rangle$ on $[a,b]$ We can find a positive integer m such that

$$\Big|s_n(x) - s(x)\Big| < \frac{\epsilon}{3(b-a)}; \;\; \forall\, n \geq m.$$

In particular, $\Big|s_m(x) - s(x)\Big| < \frac{\epsilon}{3(b-a)}$ i.e. $-\frac{\epsilon}{3(b-a)} + s_m(x) < s(x) < \frac{\epsilon}{3(b-a)} + s_m(x)$. For this fixed m, since s_m is R-integrable, we chose a partition of $[a,b]$ such that $U(P;s_m) - L(P;s_m) < \frac{\epsilon}{3}$.

$$\because \;\; s(x) < s_m(x) + \frac{\epsilon}{3(b-a)}$$
$$\therefore \;\; U(P;s) < U(P;s_m) + \frac{\epsilon}{3}$$

Again since,

$$s(x) > s_m(x) - \frac{\epsilon}{3(b-a)}$$
$$\therefore \;\; L(P;s) > L(P;s_m) - \frac{\epsilon}{3}$$

Therefore

$$U(P;s) - L(P;s) < U(P;s_m) - L(P;s_m) + \frac{2\epsilon}{3} = \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon$$

So, $s(x)$ is R-integrable on $[a,b]$. Therefore,

$$\lim_{n\longrightarrow\infty} \int_a^b s_n(x)dx = \int_a^b \Big\{\lim_{n\longrightarrow\infty} s_n(x)\Big\}dx$$
$$\sum_{n=1}^{\infty}\Big[\int_a^b u_n(x)dx\Big] = \int_a^b \Big[\sum_{n=1}^{\infty} u_n(x)\Big]dx$$

RESULT **5.** It is observed here that term-by=term integration is not a sufficient condition for uniform convergence of a series of functions as may be seen in the following example 46.

EXAMPLE **46.** *Show that the series $\sum_{n=1}^{\infty} u_n(x)$, where $u_1(x) = x$ and $u_n(x) = \Big[x^{\frac{1}{2n-1}} - x^{\frac{1}{2n-3}}\Big], n \geq 2$, is not uniformly convergent on $[0,1]$. the series be integrated term-by-term on $[0,1]$?*

Solution: The n^{th} partial sum of the series is given by

$$S_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x) = x^{\frac{1}{2n-1}}; x \in [0, 1]$$

For all $x \in (0, 1]$, $\lim_{n\to\infty} S_n(x) = 1$ and for $x = 0$, the sequence $\langle S_n \rangle$ converges to 0. Therefore, the series $\sum_{n=1}^{\infty} u_n(x)$ is convergent pointwise on $[0, 1]$ to the limit function S, where $S(x) = \begin{cases} 0; & \text{if } x = 0 \\ 1; & \text{if } x \in (0, 1] \end{cases}$. The limit function $S(x)$ is discontinuous at $x = 0$. Since each u_n is continuous on $[0, 1]$ and the limit function $S(x)$ is not continuous on $[0, 1]$, so the series $\sum u_n$ is not uniformly convergent on $[0, 1]$. Now

$$\int_0^1 \Big(\sum_{n=1}^{\infty} u_n(x)\Big)dx = \int_0^1 S(x)dx = \int_0^1 dx = 1$$
$$\int_0^1 u_1(x)dx = \int_0^1 x\ dx = \frac{1}{2}$$
$$\int_0^1 u_n(x)dx = \int_0^1 dx = \Big[x^{\dfrac{1}{2n-1}} - x^{\dfrac{1}{2n-3}}\Big] = \frac{2n-1}{2n} - \frac{2n-3}{2n-2}; n \geq 2 \qquad (i)$$

Again, let $I_n = \int_0^1 u_1(x)dx + \int_0^1 u_2(x)dx + \cdots + \int_0^1 u_n(x)dx$, then using (i), we get

$$I_n = \frac{2n-1}{2n} \Rightarrow \lim_{n\to\infty} I_n = 1$$
$$\Rightarrow \sum_{n=1}^{\infty} \int_0^1 u_n(x)dx = 1 = \int_0^1 \Big(\sum_{n=1}^{\infty} u_n(x)\Big)dx$$

i.e., the series can be integrated term-by-term on $[0, 1]$.

EXAMPLE **47.** *Prove that* $\int_0^1 \Big(\sum_{n=1}^{\infty} \frac{x^n}{n^2}\Big)dx = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$

Solution: Let $u_n(x) = \frac{x^n}{n^2}$; $x \in [0, 1]$, $n \in \mathbb{N}$. For all $x \in [0, 1]$,

$$|u_n(x)| = \left|\frac{x^n}{n^2}\right| \leq \frac{1}{n^2};\ \ \forall n \in \mathbb{N}.$$

Let $M_n = \dfrac{1}{n^2}$, then $|u_n(x)| \leq M_n$ for all $x \in [0, 1]$ and for all $n \in \mathbb{N}$ and $\sum M_n$ is a convergent series of positive real numbers.

Therefore, by Weierstrass M-test, the series $\sum u_n$ is uniformly convergent on $[0, 1]$. Since each f_n is integrable on $[0, 1]$, the series can be integrated term by term on $[0, 1]$. Hence

$$\int_0^1 \Big(\sum_{n=1}^{\infty} \frac{x^n}{n^2}\Big)dx = \sum_{n=1}^{\infty} \int_0^1 \frac{x^n}{n^2}dx = \sum_{n=1}^{\infty} \Big[\frac{x^{n+1}}{(n+1)n^2}\Big]_0^1 = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$$

EXAMPLE **48.** *Show that the series* $\sum_{n=1}^{\infty} u_n(x)$*, where* $u_n(x) = 2x\Big[\frac{1}{n^2}e^{-\frac{x^2}{n^2}} - \frac{1}{(n+1)^2}e^{-\frac{x^2}{(n+1)^2}}\Big]$*,* $x \in [0,1]$ *is uniformly convergent on* $[0,1]$ *and further show that* $\sum_{n=1}^{\infty}\int_0^1 u_n(x)dx = \int_0^1 \Big(\sum_{n=1}^{\infty} u_n(x)\Big)dx.$

Solution: The n^{th} partial sum of the series is given by

$$\begin{aligned} S_n(x) &= u_1(x) + u_2(x) + \cdots + u_n(x) \\ &= 2x\Big[e^{-x^2} - \frac{1}{(n+1)^2}e^{-\frac{x^2}{(n+1)^2}}\Big] \end{aligned}$$

For all $x \in (0,1]$, $\lim_{n\to\infty} S_n(x) = 2xe^{-x^2}$. For $x = 0$, the sequence $\langle S_n\rangle$ converges to 0. Therefore, the series $\sum_{n=1}^{\infty} u_n(x)$ converges pointwise on $[0,1]$ to the function S, where $S(x) = 2xe^{-x^2}$, $x \in [0,1]$. Therefore,

$$\int_0^1 \Big(\sum_{n=1}^{\infty} u_n(x)\Big)dx = \int_0^1 2xe^{-x^2}dx = 1 - \frac{1}{e}$$

For all $x \in [0,1]$,

$$|u_n(x)| \le 2\Big\{\frac{1}{n^2} + \frac{1}{(n+1)^2}\Big\} = M_n \text{ (say); } \forall n \in \mathbb{N}$$

Then $\sum_{n=1}^{\infty} M_n$ is a convergent series of positive real numbers and for all $x \in [0,1]$, $|u_n(x)| \le M_n$ for all $n \in \mathbb{N}$. Hence by Weierstrass M-test the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on $[0,1]$. Now

$$\begin{aligned} \int_0^1 u_n(x)dx &= \int_0^1 2x\Big[\frac{1}{n^2}e^{-\frac{x^2}{n^2}} - \frac{1}{(n+1)^2}e^{-\frac{x^2}{(n+1)^2}}\Big]dx \\ &= \Big[-e^{-\frac{x^2}{n^2}} + e^{-\frac{x^2}{(n+1)^2}}\Big]_0^1 = -e^{-\frac{1}{n^2}} + e^{-\frac{1}{(n+1)^2}} \qquad (i) \end{aligned}$$

Since each u_n is integrable on $[0,1]$ and $\sum u_n$ converges uniformly on $[0,1]$, then term-by-term integration for the series is possible and

$$\sum_{n=1}^{\infty}\int_0^1 u_n(x)dx = \int_0^1 \Big(\sum_{n=1}^{\infty} u_n(x)\Big)dx = 1 - \frac{1}{e}$$

Again, let $I_n = \int_0^1 u_1(x)dx + \int_0^1 u_2(x)dx + \cdots + \int_0^1 u_n(x)dx$, then using (i), we get

$$\begin{aligned} I_n = -\frac{1}{e} + e^{-\frac{1}{(n+1)^2}} &\Rightarrow \lim_{n\to\infty} I_n = 1 - \frac{1}{e} \\ \Rightarrow \sum_{n=1}^{\infty}\int_0^1 u_n(x)dx &= 1 - \frac{1}{e} \end{aligned}$$

EXAMPLE **49.** *Let us consider the series* $\sum_{n=1}^{\infty} u_n(x)$, *where* $u_n(x) = x\Big[n^2 e^{-n^2x^2} - (n-1)^2 e^{-(n-1)^2x^2}\Big]$, $x \in [0,1]$. *Applying integration show that the series* $\sum_{n=1}^{\infty} u_n(x)$ *is not uniformly convergent on* $[0,1]$.

Solution: The n^{th} partial sum of the series is given by

$$S_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x) = n^2 x e^{-n^2x^2}$$

For all $x \in (0,1]$,

$$e^{n^2x^2} > \frac{n^4x^4}{2} > 0 \Rightarrow 0 < S_n(x) < \frac{2}{n^3x^3}; \quad \forall x \in (0,1].$$

Thus, by Sandwich theorem $\lim_{n\to\infty} S_n(x) = 0$ for all $x \in (0,1]$. For $x = 0$, the sequence $\langle S_n \rangle$ converges to 0. Therefore, the series $\sum_{n=1}^{\infty} u_n(x)$ converges pointwise on $[0,1]$ to the function S, where $S(x) = 0$, $x \in [0,1]$. Therefore,

$$\int_0^1 \Big(\sum_{n=1}^{\infty} u_n(x)\Big) dx = \int_0^1 0 \, dx = 0$$

Now

$$\begin{aligned} \int_0^1 u_n(x)dx &= \int_0^1 x\Big[n^2 e^{-n^2x^2} - (n-1)^2 e^{-(n-1)^2x^2}\Big]dx \\ &= \frac{1}{2}\Big[e^{-(n-1)^2x^2} - e^{n^2x^2}\Big] = \frac{1}{2}\Big[e^{-(n-1)^2} - e^{n^2}\Big] \qquad (i) \end{aligned}$$

Again, let $I_n = \int_0^1 u_1(x)dx + \int_0^1 u_2(x)dx + \cdots + \int_0^1 u_n(x)dx$, then using (i), we get

$$\begin{aligned} I_n &= \frac{1}{2}\Big(1 - e^{-n^2}\Big) \Rightarrow \lim_{n\to\infty} I_n = \frac{1}{2} \\ \Rightarrow \quad \sum_{n=1}^{\infty} \int_0^1 u_n(x)dx &= \frac{1}{2} \neq \int_0^1 \sum_{n=1}^{\infty} u_n(x)dx \end{aligned}$$

Therefore, the series $\sum_{n=1}^{\infty} u_n(x)$ is not uniformly convergent on $[0,1]$.

EXAMPLE **50.** *If* $S(x)$ *be the sum function of the series* $\sum_{n=1}^{\infty} u_n(x)$, *where* $u_n(x) = ne^{-nx}$, $x \in [a,b]$, $0 < a < b$, *then show that the series converges uniformly to* $S(x)$ *on* $[a,b]$. *Evaluate* $\int_{\log 2}^{\log 3} S(x)dx$.

Solution: For all $x \in [a,b]$

$$|u_n(x)| = \frac{n}{e^{nx}} < \frac{2n}{n^2x^2} < \frac{2}{n^2a^2};\ \ \forall n \in \mathbb{N}$$

Let $M_n = 2/a^2n^2$, then $\sum u_n(x)$ converges uniformly on $[a,b]$ where $0 < a < b$ to the sum function $S(x)$, where

$$S(x) = \sum_{n=1}^{\infty} u_n(x);\ \ x \in [a,b], 0 < a < b.$$

Now, each u_n is integrable on $[a,b]$. Also, the series $\sum u_n$ is uniformly convergent on $[a,b]$ to the sum function $S(x)$. Therefore

$$\begin{aligned}
\int_a^b S(x)dx &= \sum_{n=1}^{\infty}\int_a^b u_n(x)dx = \int_a^b u_1(x)dx + \int_a^b u_2(x)dx + \cdots \\
\int_{\log 2}^{\log 3} S(x)dx &= \int_{\log 2}^{\log 3} e^{-x}dx + \int_{\log 2}^{\log 3} 2e^{-2x}dx + \int_{\log 2}^{\log 3} 3e^{-2x}dx + \cdots \\
&= \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{2^2}-\frac{1}{3^2}\right) + \left(\frac{1}{2^3}-\frac{1}{3^3}\right) + \cdots \\
&= \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right) - \left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots\right) \\
&= 2 - \frac{3}{2} = \frac{1}{2}
\end{aligned}$$

4.4 Uniform Convergence and Differentiation

Here, we look at the differentiability properties of the limit of a uniformly convergent sequence of differentiable functions. Here, the situation is more complicated. In fact, even the uniform limit of a sequence of differentiable functions need not be differentiable.

Consider the sequence $\langle f_n(x)\rangle$ where $f_n(x) = xe^{-nx^2}$, $-1 \le x \le 1$. Now

$$f(x) = \lim_{n\longrightarrow\infty} s_n(x) = x \lim_{n\longrightarrow\infty} e^{-nx^2} = 0.$$

$\therefore$ $\langle f_n(x)\rangle$ converges uniformly to $f(x) = 0$ for all values of x in $[-1,1]$. Hence $f'(x) = 0$ $\forall\, x \in [-1,1]$, so $f'(0) = 0$. But

$$f_n'(x) = e^{-nx^2} + x(-2nx)e^{-nx^2} = e^{-nx^2}(1-2nx^2).$$

At $x = 0$, $s_n'(0) = 1$; $\forall\, n$. Therefore, $f_n'(0) \longrightarrow 1$ as $n \longrightarrow \infty$ and $f'(0) = 0$. So uniform convergence of $\langle f_n(x)\rangle$ is not enough to guarantee that $\frac{d}{dx}\left\{\lim_{n\longrightarrow\infty} f_n(x)\right\} = \lim_{n\longrightarrow\infty}\left\{\frac{d}{dx}f_n(x)\right\}$.

THEOREM **16.** *Let $\langle f_n\rangle$ be a real valued function defined on $[a,b]$ such that,*

(i) each f_n is continuously differentiable function on $[a,b]$

(ii) $\langle f_n(x)\rangle$ converges at least at one point $x_0 \in [a,b]$

(iii) $\langle f_n'\rangle$ converges uniformly to a function $\sigma(x)$ on $[a,b]$.

Then

1. $\langle f_n(x)\rangle$ *must converges uniformly to a continuously differentiable function* $f(x)$ *on* $[a,b]$*, and*
2. $f'(x) = \sigma(x)$ *i.e.,* $\frac{d}{dx}\left\{\lim_{n\longrightarrow\infty} f_n(x)\right\} = \lim_{n\longrightarrow\infty}\left\{\frac{d}{dx}f_n(x)\right\}; \forall x \in [a,b]$.

Proof: Let $\epsilon > 0$ be chosen number. Since $\langle f_n'\rangle$ converges uniformly on $[a,b]$, so, $\exists$ a positive integer $N_1(\epsilon) \in \mathbb{N}$ such that

$$\left|f_n'(x) - f_m'(x)\right| < \frac{\epsilon}{2(b-a)}\ ;\ \forall\, m,n \geq N_1,\ \forall\, x \in [a,b]$$

Also since $\langle f_n(x)\rangle$ converges at $x = x_0$, corresponding to the same ϵ, $\exists$ a natural number $N_2(\varepsilon) \in \mathbb{N}$ such that

$$\left|f_n(x_0) - f_m(x_0)\right| < \frac{\epsilon}{2}\ ;\ \forall\, m,n \geq N_2$$

Let x and y be any points in $[a,b]$. Since $f_n(x)$ is differentiable and hence continuous on $[a,b]$, by using Lagrange mean value theorem, we get

$$\begin{aligned}
&\left|\{f_n(x) - f_m(x)\} - \{f_n(y) - f_m(y)\}\right| \\
&= \left|(x-y)\{f_n'(\xi) - f_m'(\xi)\}\right|;\quad \text{where, } \xi \in (x,y) \\
&< (b-a)\cdot\frac{\epsilon}{2(b-a)} < \frac{\epsilon}{2};\quad \text{as } |x-y| < b-a
\end{aligned}$$

Let, $N = \max\{N_1, N_2\} \in \mathbb{N}$. Then, for $\forall\, m,n \geq N$ and for all $x \in [a,b]$, we have

$$\begin{aligned}
|f_n(x) - f_m(x_0)| &= |\{f_n(x) - f_m(x)\} - \{f_n(x_0) - f_m(x_0)\} + \{f_n(x_0) - f_m(x_0)\}| \\
&\leq \left|\{f_n(x) - f_m(x)\} - \{f_n(x_0) - f_m(x_0)\}\right| + |\{f_n(x_0) - f_m(x_0)\}| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \varepsilon
\end{aligned}$$

Therefore, by Cauchy criterion, $\langle f_n\rangle$ converges uniformly on $[a,b]$ and f be the uniform limit of $\langle f_n\rangle$ on $[a,b]$.

For fixed x on $[a,b]$, and for any $y \in [a,b]$; let us define

$$\phi_n(y) = \frac{f_n(y) - f_n(x)}{y-x}; n \in \mathbb{N} \text{ and } \phi(y) = \frac{f(y) - f(x)}{y-x}; y \neq x$$

Since each f_n is differentiable on $[a,b]$, so for each $n \in \mathbb{N}$

$$\lim_{y\rightarrow x}\phi_n(y) = \lim_{y\rightarrow x}\frac{f_n(y) - f_n(x)}{y-x} = f_n'(x).$$

Now, for all $m,n \geq N$, we have

$$\begin{aligned}
|\phi_n(y) - \phi_m(y)| &= \frac{1}{|y-x|}\cdot\left|-\{f_n(x) - f_m(x)\} + \{f_n(y) - f_m(y)\}\right| \\
&< \frac{1}{y-x}\cdot|x-y|\cdot\frac{\varepsilon}{2(b-a)} = \frac{\varepsilon}{2(b-a)}.
\end{aligned}$$

Therefore, $\{\phi_n(y)\}$ converges uniformly to $\phi(y)$ on $[a,b]$ for $y \in [a,b]$ but $y \neq x$. Since $\{f_n(x)\}_n$ converges uniformly to $f(x)$ on $[a,b]$, we get

$$\begin{aligned} & \lim_{y\to x}\{\lim_{n\to\infty}\phi_n(y)\} = \lim_{n\to\infty}\{\lim_{y\to x}\phi_n(y)\} \\ \text{or,}\quad & \lim_{y\to x}\phi(y) = \lim_{n\to\infty} f_n'(x) \\ \text{or,}\quad & \lim_{y\to x}\frac{f(y)-f(x)}{y-x} = \lim_{n\to\infty} f_n'(x) \\ \text{or,}\quad & f_n'(x) = \sigma(x); \forall x \in [a,b] \end{aligned}$$

But $f'(x) = \frac{d}{dx}\{f(x)\} = \frac{d}{dx}\left\{\lim_{n\to\infty} f_n(x)\right\}$ and $\sigma(x) = \lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty}\frac{d}{dx}f_n(x)$. So,

$$\frac{d}{dx}\left\{\lim_{n\to\infty} f_n(x)\right\} = \lim_{n\to\infty}\frac{d}{dx}f_n(x);\quad \forall x \in [a,b]$$

This completes the proof of the theorem. □

EXAMPLE **51.** Show that for the series whose partial sums are given by $s_n(x) = n^2xe^{-n^2x^2}$

(i) the limit function is continuous.

(ii) term-by-term integration is valid

(iii) term-by-term differentiation is valid but

(iv) the series does not converge uniformly on any closed interval containing origin

Solution: Let $[a,b]$ be a closed and bounded interval containing 0. Here $s(x) = \lim_{n\to\infty} s_n(x) = 0 \forall x \in [a,b]$, a constant function. So, s is continuous on $[a,b]$.
(ii)

$$\begin{aligned} & \int_a^b \left\{\lim_{n\to\infty} s_n(x)\right\}dx = \int_a^b 0dx = 0. \\ & \int_a^b s_n(x)dx = \int_a^b n^2xe{-}n^2x^2dx = -\frac{1}{2}\left\{e^{-n^2b^2} - e^{-n^2a^2}\right\}[\text{ Put } n^2x^2 = u] \\ \therefore\quad & \lim_{n\to\infty}\int_a^b s_n(x)dx = 0 \end{aligned}$$

So, $\int_a^b \{\sum f_n(x)\}dx = \sum \int_a^b f_n(x)dx$. Thus, term-by-term integration is valid.
(iii) Since $s(x) = 0; \forall x \in [a,b]$. $\therefore s'(x) = 0 \forall x \in [a,b]$. Now, $s_n'(x) = n^2e^{-n^2x^2}(1-2n^2x^2)$. Therefore

$$\begin{aligned} & \lim_{n\to\infty} s_n'(x) = 0;\quad x \in [a,b] \\ & \lim_{n\to\infty} s_n'(x) = s'(x),\ \text{i.e.,}\ \frac{d}{dx}\left\{\lim_{n\to\infty} s_n(x)\right\} = \lim_{n\to\infty}\frac{d}{dx}(s_n(x)) \\ \text{or,}\quad & \frac{d}{dx}\left\{\sum_{n=0}^{\infty} f_n(x)\right\} = \sum_{n=0}^{\infty} f_n'(x);\quad \forall x \in [a,b] \end{aligned}$$

Thus term-by-term differentiation is valid for all $x \in [a, b]$.

(iv) $M_n = \sup_{x \in [a,b]} |s_n(x) - s(x)| = \sup_{x \in [a,b]} \left|\frac{n^2 x}{e^{n^2} x^2}\right| = |g(x)|$, say. Now

$$g'(x) = x^2 e^{-n^2x^2}(1 - 2n^2x^2) = 0 \text{ for } x = \pm\frac{1}{n\sqrt{2}}$$
$$g''(x) = n^4 x(4n^2x^2 - 6)e^{n^2x^2} < 0 \text{ at } x = \frac{1}{n\sqrt{2}}$$

Therefore, $M_n = \frac{1}{2}ne^{-1/2} \to \infty$ as $n \to \infty$. So the series is not uniformly convergent on $[a, b]$, an interval containing 0.

THEOREM **17** (Term-by-Term Differentiation)**.** *Let $\langle u_n \rangle$ be a sequence of differentiable functions on $[a, b]$, such that the series $\sum_{n=1}^{\infty} u_n(x_0)$ converges for some $x_0 \in [a, b]$. If the series $\sum_{n=1}^{\infty} u'_n$ of derivatives converges uniformly on $[a, b]$, then the series $\sum_{n=1}^{\infty} u_n$ converges uniformly on $[a, b]$ to a differentiable sum s and we have*

$$s'(x) = \frac{d}{dx}\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} u'_n(x); \;\; \forall x \in [a, b]$$

Proof: The n^{th} partial sum of the series $\sum_{n=1}^{\infty} u_n$ is $s_n(x) = u_1(x) + u_2(x) + u_3(x) + \cdots + u_n(x)$. Now,

(i) Each u_n is given to be differentiable on $[a, b]$ and so s_n is also differentiable on $[a, b]$, and $s'_n(x) = u_1{}'(x) + u_2{}'(x) + u_3{}'(x) + \cdots + u_n{}'(x)$.

(ii) If $\sum_{n=1}^{\infty} u'_n$ be the series of derived functions then this $s'_n(x) = \sum_{n=1}^{\infty} u'_n(x)$

(iii) Given that the series $\sum_{n=1}^{\infty} u'_n(x)$ converges uniformly to $\sigma(x)$ on $[a, b]$, so we may take $\{u'_n(x)\}_n$ converges to $\sigma(x)$ on $[a, b]$.

(iv) Given that the series $\sum_{n=1}^{\infty} u'_n(x)$ converges at least one point $x_0 \in [a, b]$, so we can take $\{s'_n(x)\}_n$ converges at least one point $x_0 \in [a, b]$.

Hence, by the previous theorem 16, the sequence $\{s_n(x)\}$ must converge uniformly to its limit function $s(x)$ on $[a, b]$ such that $s'(x) = \sigma(x); \forall x \in [a, b]$.

EXAMPLE **52.** We end this section by giving an example of a continuous function on R that is nowhere differentiable

(i) Consider the *sawtooth function*:

$$f_0(x) = \begin{cases} x - [x]; & \text{if } x \le [x] + \frac{1}{2} \\ [x] + 1 - x; & \text{if } x > [x] + \frac{1}{2} \end{cases}$$

Then $f_0(x)$ is the distance from x to the nearest integer, i.e., $f_0(x) = d(x, \mathbb{Z})$, and is a continuous, periodic function on $\mathbb{R}$ with period 1. Now, define $f_n(x) = 4^{-n} f_0(4^n x)$ for all $x \in \mathbb{R}$ and $n = 0, 1, 2, \cdots$. Then f_n is also a continuous sawtooth function (with period 4^{-n}), whose graph consists of line segments of slope ± 1. Since $0 \leq f_0 \leq \frac{1}{2}$, we have $0 \leq f_n(x) \leq \dfrac{1}{2 \cdot 4^n}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

RESULT **6.** Only the uniform convergence of the series of functions $\sum_n u_n(x)$ on $[a, b]$ is not sufficient to ensure the validity of term-by-term differentiation of the series $\sum_n u_n(x)$ on $[a, b]$. This situation is depicted in the following example 53.

EXAMPLE **53.** example We consider the series $\displaystyle\sum_{n=1}^{\infty} u_n(x)$, $x \in [0, 1]$ whose n^{th} partial sum is $s_n(x) = \dfrac{x}{1 + nx^2}, x \in [0, 1]$. Then $\lim_{n\to\infty} s_n(x) = 0$ for all $x \in [0, 1]$. Hence, the sequence $\langle s_n(x) \rangle$ converges pointwise to the limit function $s(x)$ where $s(x) = 0$, $x \in [0, 1]$. Let (as depicted in Example 20)

$$M_n = \sup_{x\in[0,1]} \Big| s_n(x) - s(x) \Big| = \sup_{x\in[0,1]} \frac{x}{1 + nx^2} = \frac{1}{2\sqrt{n}} \to 0 \text{ as } n \to \infty$$

Therefore $\langle s_n(x) \rangle$ is uniformly convergent on $[0, 1]$. Thus, the series $\sum_n u_n(x)$ converges uniformly to the limit function f on $[0, 1]$. Now

$$s'_n(x) = \frac{d}{dx}\Big\{ s_n(x) \Big\} = \frac{1 - nx^2}{(1 + nx^2)^2} \text{ and } \lim_{n\to\infty} s'_n(x) = \begin{cases} 0; & 0 < x \leq 1 \\ 1; & x = 0 \end{cases}$$

Hence the series $\sum u'_n$ converges to the function g, where $g(x) = \begin{cases} 0; & 0 < x \leq 1 \\ 1; & x = 0 \end{cases}$. Therefore

$$\begin{aligned} &\frac{d}{dx} u_1(x) + \frac{d}{dx} u_2(x) + \cdots = 0 = \frac{d}{dx}\Big[u_1(x) + u_2(x) + \cdots \Big], \text{ for } 0 < x \leq 1 \\ \text{and} \quad &\frac{d}{dx} u_1(x) + \frac{d}{dx} u_2(x) + \cdots = 1 \neq \frac{d}{dx}\Big[u_1(x) + u_2(x) + \cdots \Big], \text{ for } x = 0 \end{aligned}$$

RESULT **7.** If the series $\sum s_n$ be convergent pointwise, then the uniform convergence of the series $\sum s'_n$ is only a sufficient condition for the validity of term-by-term differentiation of the series $\sum s_n$. This situation is depicted in the following example 54.

EXAMPLE **54.** We consider the series $\displaystyle\sum_{n=1}^{\infty} u_n(x)$, $x \in [0, 1]$ whose n^{th} partial sum is $s_n(x) = \dfrac{\log(1 + n^4 x^2)}{2n^2}, x \in [0, 1]$. Then

$$\lim_{n\to\infty} s_n(x) = \lim_{n\to\infty} \frac{\log(1 + n^4 x^2)}{n^2} = 0 \text{ for all } x \in [0, 1]$$

Therefore, the sequence $\langle s_n(x) \rangle$ converges pointwise to the limit function $s(x)$ where $s(x) = 0$, $x \in [0, 1]$. Now

$$s'_n(x) = u'_1(x) + u'_2(x) + \cdots + u'_n(x) = \frac{n^2 x}{1 + n^4 x^2}; x \in [0, 1]$$

Therefore, $\lim_{n\to\infty} s'_n(x) = 0$ for all $x \in [0,1]$. Thus the sequence $\langle s'_n(x)\rangle$ converges point wise to the limit function $s(x)$ where $s(x) = 0$, $x \in [0,1]$ and hence the series $\sum s'_n(x)$ converges to the function $g(x)$ on $[0,1]$. Now $\frac{d}{dx}[f(x)] = 0$ for all $x \in [0,1]$ and also $\frac{d}{dx}[f(x)] = g(x), x \in [0,1]$. Therefore,

$$\frac{d}{dx}[u_1(x)] + \frac{d}{dx}[u_2(x)] + \cdots = \frac{d}{dx}\Big[u_1(x) + u_2(x) + \cdots\Big]$$

Thus term-by-term differentiation of the series $\sum u_n$ is valid. Let

$$M_n = \sup_{x\in[0,1]} \Big|s'_n(x) - s(x)\Big| = \sup_{x\in[0,1]} \frac{n^2x}{1+n^4x^2}$$

For $x > 0$, we have $\frac{n^2x + \frac{1}{n^2x}}{2} \geq \sqrt{n^2x \cdot \frac{1}{n^2x}}$, equality holds for $x = \frac{1}{n^2}$. Therefore, $\frac{n^2x}{1+n^4x^2} \leq \frac{1}{2}$ for all $x > 0$, equality holds for $x = \frac{1}{n^2}$. Again for $x = 0$, $\frac{n^2x}{1+n^4x^2} = 0$. Hence

$$M_n = \sup_{x\in[0,1]} \frac{n^2x}{1+n^4x^2} = \frac{1}{2}$$

Since $\lim_{n\to\infty} M_n = \frac{1}{2}(\neq 0)$, the sequence $\langle s'_n(x)\rangle$ and hence the series $\sum u'_n$ is not uniformly convergent on $[0,1]$. Thus, although the series $\sum u'_n$ is not uniformly convergent on $[0,1]$, term by term differentiation of the series $\sum u_n$ is valid.

EXAMPLE **55.** *Show that term-by-term differentiation is not valid at $x = 0$ for the series $\sum_{n=1}^{\infty} u_n(x)$, where $u_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}; x \in [0,1]$.*

Solution: Let $s_n(x)$ be the n^{th} partial sum of the series $\sum u_n(x)$, then

$$s_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x) = \frac{nx}{1+n^2x^2}; \quad x \in [0,1]$$

Therefore, $\lim_{n\to\infty} s_n(x) = 0$ for all $x \in [0,1]$ and hence the sequence $\langle s_n(x)\rangle$ converges pointwise to the limit function $s(x)$, where $s(x) = 0$, $x \in [0,1]$. Thus the series $\sum u_n(x)$ converges pointwise to the limit function $s(x)$ for all $x \in [0,1]$. Now

$$\frac{d}{dx}\Big(\sum u_n(x)\Big) = \frac{d}{dx}\Big(\Big) = 0; \quad \forall x \in [0,1]$$

$$\text{and} \qquad \frac{d}{dx}\Big(u_n(x)\Big) = \frac{n - n^3x^2}{\Big(1+n^2x^2\Big)^2} - \frac{(n-1)-(n-1)^3x^2}{\Big[1+(n-1)^2x^2\Big]^2}$$

At $x = 0$, $\frac{d}{dx}\Big(u_n(x)\Big) = n - (n-1) = 1$ and hence $\sum \frac{d}{dx}\Big(u_n(x)\Big) = 1 + 1 + \cdots$, which is a divergent series. Therefore, at $x = 0$, $\frac{d}{dx}\Big(\sum u_n(x)\Big) \neq \sum \frac{d}{dx}\Big(u_n(x)\Big)$.

Problem Set

[Multiple Choice Questions]

1. Let $\lim_{n\to\infty} x^n = f(x)$, $x \in [0,1]$. Then

   a) $f(x) = \begin{cases} 0; & \text{if } 0 \le x < 1 \\ 1; & \text{if } x = 1 \end{cases}$
   b) $f(x) = \begin{cases} 1; & \text{if } 0 \le x < 1 \\ 0; & \text{if } x = 0 \end{cases}$

   c) $f(x) = 1, x \in [0,1]$
   d) $f(x) = 0; x \in [0,1]$

   1 (a)

2. The series $\sum_{n=1}^{\infty} x^n$, converges pointwise to the sum function $s(x)$, $x \in [0,1]$. Then

   a) $f(x) = \frac{1}{1-x}, x \in [0,1]$
   b) $f(x) = \frac{1}{1+x}, x \in [0,1]$

   c) $f(x) = x, x \in [0,1]$
   d) $f(x) = 0$ does not exist

   2 (d)

3. For each $n \in \mathbb{N}$, let $f_n(x) = \lim_{m\to\infty} \Big(\cos n!\pi x\Big)^{2m}; x \in \mathbb{R}$. Then the sequence of functions $\langle f_n \rangle$ converges on $\mathbb{R}$ to the function f defined by

   a) $f(x) = \begin{cases} 0; & \text{if } x \in \mathbb{Q} \\ 1; & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$
   b) $f(x) = \begin{cases} 1; & \text{if } x \in \mathbb{Q} \\ 0; & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$

   c) $f(x) = \begin{cases} \pi; & \text{if } x \in \mathbb{Q} \\ 1; & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$
   d) $f(x) = \begin{cases} -\pi; & \text{if } x \in \mathbb{Q} \\ 1; & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$

   3 (b)

4. Which of the following sequences of functions is uniformly convergent on $(0,1)$?

   a) x^n
   b) $\frac{n}{nx+1}$
   c) $\frac{x}{nx+1}$
   d) $\frac{1}{nx+1}$

   4 (c)

5. Let $f_n(x) = x^{1/n}$ for $x \in [0,1]$. Then

   a) $\lim_{n\to\infty} f_n(x)$ exists for all $x \in [0,1]$
   b) $\lim_{n\to\infty} f_n(x)$ defines a continuous function on $[0,1]$

   c) $\{f_n(x)\}$ converges uniformly on $[0,1]$
   d) $\lim_{n\to\infty} f_n(x) = 0$ for all $n \in [0,1]$

   5 (a)

6. Let $\lim_{n\to\infty} xe^{-nx} = f(x)$, $x \ge 0$. Then

   a) $f(x) = 0, x \ge 0$
   b) $f(x) = 1, x \ge 0$
   c) $f(x) = e^{-1}, x \ge 0$
   d) none of the above

6 (a)

7. Let $f_n(x) = \tan^{-1} nx; x \in \mathbb{R}$. Then $\lim\limits_{n\to\infty} f_n(x)$ is

a) 0 b) $\dfrac{\pi}{2}$ c) $-\dfrac{\pi}{2}$ d) $\dfrac{\pi}{2} sgnx$

7 (d)

8. For $x \in (-1, 1)$, the sum of the series $\dfrac{1}{1+x} + \dfrac{2x}{1+x^2} + \dfrac{4x^3}{1+x^4} + \cdots + \dfrac{2^n x^{2^n-1}}{1+x^{2^n}} + \cdots$ is

a) $\dfrac{1}{1+x}$ b) $\dfrac{1}{1-x}$ c) 0 d) $\dfrac{1-x}{1+x}$

8 (b)

9. The series $1 - \dfrac{e^{-2x}}{2^2-1} + \dfrac{e^{-4x}}{4^2-1} - \dfrac{e^{-6x}}{6^2-1} + \cdots$ is

a) converges uniformly for all $x \geq 0$
b) converges uniformly for all $x \in \mathbb{R}$
c) converges uniformly for all $x \in (-1, 1)$
d) converges uniformly on $[-1, 1]$

9 (a)

10. Which of the following sequence $\langle f_n \rangle$ of functions does not converge uniformly on $[0, 1]$?

a) $f_n(x) = \dfrac{e^{-x}}{n}$
b) $f_n(x) = (1-x)^n$
c) $f_n(x) = \dfrac{x^2+nx}{n}$
d) $f_n(x) = \dfrac{\sin(nx+n)}{n}$

10 (b)

11. Which one of the following series of functions is uniformly convergent for all real x ?

a) $\sum \dfrac{(-1)^n x^{2n}}{\sqrt{n}(1+x^{2n})}$ b) $\sum \dfrac{(-1)^n x^{2n}}{n^{3/2}(1+x^{2n})}$ c) $\sum \dfrac{(-1)^n x^{2n}}{n^{2/3}(1+x^{2n})}$ d) None of these

11 (b)

12. Let $f_n(x) = \begin{cases} 1 - nx; & \text{for } 0 \leq x \leq \frac{1}{n} \\ 0; & \text{for } \frac{1}{n} < x \leq 1 \end{cases}$

a) $\lim\limits_{n\to\infty} f_n(x)$ defines a continuous function on $[0, 1]$
b) $\{f_n\}$ converges uniformly on $[0, 1]$
c) $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \in [0, 1]$
d) $\lim\limits_{n\to\infty} f_n(x)$ exists for all $x \in [0, 1]$

12 (d)

13. Let $f_n(x) = \begin{cases} 1 - nx; & \text{for } 0 \leq x \leq \frac{1}{n} \\ 0; & \text{for } \frac{1}{n} < x \leq 1 \end{cases}$ and $\lim\limits_{n\to\infty} f_n = f$. Then

a) f is continuous on $[0,1]$
b) f is bounded on $[0,1]$
c) convergence of $\langle f_n \rangle$ is uniform
d) none of these

13 (b)

14. Let $f_n(x) = \begin{cases} n(1-nx); & \text{for } 0 \leq x \leq \frac{1}{n} \\ 0; & \text{for } \frac{1}{n} < x \leq 1 \end{cases}$. Then $\langle f_n \rangle$

a) converges pointwise to $f(x) = 0, x \in [0,1]$
b) converges uniformly to $f(x) = 0, x \in [0,1]$
c) $\lim\limits_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$
d) none of the above

14 (a)

15. Let $f_n(x) = \dfrac{x^n}{1+x^n}; x \in [0,3]$. Then

a) Convergence of $\langle f_n \rangle$ is uniform on $[0,3]$
b) The limit function is continuous on $[0,3]$
c) the limit function is bounded on $[0,3]$
d) The convergence is not pointwise on $[0,3]$

15 (c)

16. The series $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{1+x^n}$ of functions converges

a) uniformly on $[0,1]$
b) pointwise on $[0,1]$
c) sum function on $[0,1]$
d) all of these

16 (a)

17. Let $f_n : [1,2] \to [0,1]$ be given by $f_n(x) = (2-x)^n$ for all non-negative integers n. Let $f(x) = \lim\limits_{n\to\infty} f_n(x)$ for $1 \leq x \leq 2$. Then which of the following is true?

a) f is a continuous function on $[1,2]$
b) f_n converges uniformly to f on $[1,2]$ as $n \to \infty$
c) $\lim\limits_{n\to\infty} \int_1^2 f_n(x)dx = \int_1^2 f(x)dx$
d) For any $a \in (1,2)$ we have $\lim\limits_{n\to\infty} f_n'(a) \neq f'(a)$

17 (c)

18. Let $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers. Then a necessary and sufficient condition for the sequence of polynomials $f_n(x) = b_n x + c_n x^2$ to converges uniformly to 0 on the real line is

a) $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} c_n = 0$

b) $\sum_{n=1}^{\infty} |b_n| < \infty$ and $\sum_{n=1}^{\infty} |c_n| < \infty$

c) There exists a positive integer N such that $b_n = 0$ and $c_n = 0$ for all $n > N$

d) $\lim_{n\to\infty} c_n = 0$

18 (c)

19. Which one of the statement is true for the sequence of functions: $f_n(x) = \dfrac{1}{n^2+x^2}, n = 1, 2, \ldots, x \in [1/2, 1]$?

a) The sequence is monotonic and has 0 as the limit for all $x \in [1/2, 1]$ as $n \to \infty$

b) The sequence is not monotonic but has $f(x) = \dfrac{1}{x^2}$ as the limit as $n \to \infty$

c) The sequence is monotonic and has $f(x) = \dfrac{1}{x^2}$ as the limit as $n \to \infty$

d) The sequence is not monotonic but has 0 as the limit

19 (a)

20. For $n \geq 1$, let $f_n(x) = xe^{-nx^2}, x \in \mathbb{R}$. Then the sequence $\{f_n\}$ is

a) Uniformly convergent on $\mathbb{R}$

b) Uniformly convergent only on compact subset of $\mathbb{R}$

c) Bounded and not uniformly convergent on $\mathbb{R}$

d) A sequence of unbounded functions

20 (a)

21. Let $f_n(x) = n\sin^{2n+1} x \cos x$. Then the value of $\lim_{n\to\infty} \int_0^{\pi/2} f_n(x)dx - \int_0^{\pi/2} \lim_{n\to\infty} f_n(x)dx$ is

a) $\frac{1}{2}$

b) 0

c) $-\frac{1}{2}$

d) $-\infty$

21 (a)

22. Let $\{f_n\}$ be a sequence of continuous real-valued functions defined on $[0, \infty)$. Suppose $f_n(x) \to f(x)$ for all $x \in [0, \infty)$ and that f is integrable. Then

a) $\int_0^{\infty} f_n(x)dx \to \int_0^{\infty} f(x)dx$ as $n \to \infty$

b) If $f_n \to f$ uniformly on $[0, \infty)$,then $\int_0^1 f_n(x)dx \to \int_0^1 f(x)dx$

c) If $f_b \to f$ uniformly on $[0, \infty)$, then $\int_0^{\infty} f_n(x)dx \to \int_0^{\infty} f(x)dx$

d) If $\int_0^1 |f_n(x) - f(x)| \to 0$, then $f_n \to f$ uniformly on $[0, 1]$

22 (b)

23. Let $f : \mathbb{R} \to \mathbb{R}$ be strictly increasing continuous function. If $\langle a_n \rangle$ is a sequence in $[0, 1]$, then the sequence $\langle f(a_n) \rangle$ is

a) increasing b) bounded c) convergent d) not necessarily bounded

23 (b)

24. The series $\sum_{n=1}^{\infty} \frac{1}{n^2 + n^3 x^2}$ converges uniformly for

a) all $x \in \mathbb{R}$ b) $x \geq 0$ c) $x \in [0, 1]$ d) $[a, \infty), a > 0$

24 (a)

25. Which of the following conditions below imply that a function $f : [0, 1] \to \mathbb{R}$ is necessarily of bounded variation?

a) f is a monotone function on $[0, 1]$

b) f is a continuous and monotone function on $[0, 1]$

c) f has a derivative at each $x \in (0, 1)$

d) f has a bounded derivative on the interval $(0, 1)$

25 (a), (b), (c)

26. Let $f : \mathbb{R} \to [\nvdash, \infty)$ be a non-negative real valued continuous function. Let $\phi_n(x) = \begin{cases} n & \text{if } f(x) \geq n \\ 0 & \text{if } f(x) < n \end{cases}$, $\phi_{n,k}(x) = \begin{cases} \frac{k}{2^n} & \text{if } f(x) \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) \\ 0 & \text{if } f(x) \notin \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) \end{cases}$ and $g_n(x) = \phi_n(x) + \sum_{k=0}^{n2^n - 1} \phi_{n,k}(x)$. As $n \uparrow \infty$, which of the following are true?

a) $g_n(x) \uparrow f(x)$ for every $x \in \mathbb{R}$

b) Given any $C > 0, g_n(x) \uparrow f(x)$ uniformly on the set $\{x : f(x) < C\}$

c) $g_n(x) \uparrow f(x)$ uniformly for $x \in \mathbb{R}$

d) Given any $C > 0, g_n(x) \uparrow f(x)$ uniformly on the set $\{x : f(x) \geq C\}$

26 (a), (b)

27. Let $A_n \subseteq \mathbb{R}$ for $n \geq 1$, and $\chi_n : \mathbb{R} \to \{0, 1\}$ be the function $\chi_n(x) = \begin{cases} 0 & \text{if } x \notin A_n \\ 1 & \text{if } x \in A_n \end{cases}$.
Let $g(x) = \lim_{n \to \infty} sup \chi_n(x)$ and $h(x) = \lim_{n \to \infty} \chi_n(x)$

a) If $g(x) = h(x) = 1$,then there exists m such that for all $n \geq m$ we have $x \in A_n$

b) If $g(x) = 1$ and $h(x) = 0$. then there exist m such that for all $n \geq m$ we have $x \in A_n$

c) If $g(x) = 1$ and $h(x) = 0$ then there exists a sequence $n_1, n_2, \ldots$ of distinct integers such that $x \in A_{n_k}$ for all $k \geq 1$

d) If $g(x) = h(x) = 0$ then there exists m such that for all $n \geq m$ we have $x \notin A_n$

27 (a), (c), (d)

28. Let $\{f_n\}$ be a sequence of continuous functions on $\mathbb{R}$

a) If $\{f_n\}$ converges to f pointwise on $\mathbb{R}$ then $\lim\limits_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty} f(x)dx$

b) If $\{f_n\}$ converges to f uniformly on $\mathbb{R}$ then $\lim\limits_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty} f(x)dx$

c) If $\{f_n\}$ converges to f uniformly on $\mathbb{R}$ then f is continuous on $\mathbb{R}$.

d) There exists a sequence of continuous functions $\{f_n\}$ on $\mathbb{R}$, such that $\{f_n\}$ converges to f uniformly on $\mathbb{R}$, but $\lim\limits_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx \neq \int_{-\infty}^{\infty} f(x)dx$

28 (c), (d)

29. For $n \geq 1$, let $g_n(x) = \sin^2(x = \frac{1}{n}, x \in [0, \infty)$ and $f_n(x) = \int\limits_0^x g_n(t)dt$. Then

a) $\{f_n\}$ converges pointwise to a function f on $[0, \infty)$, but does not converge uniformly on $[0, \infty)$

b) $\{f_n\}$ does not converge pointwise to nay function on $[0, \infty)$

c) $\{f_n\}$ converges uniformly on $[0, 1]$

d) $\{f_n\}$ converges uniformly on $[0, \infty)$

29 (c), (d)

30. Let t and a be positive real numbers. Define $B_a = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n | x_1^2 + x_2^2 + \cdots + x_n^2 \leq a^2\}$. Then for any compactly supported continuous function f on $\mathbb{R}^n$ which of the following are correct?

a) $\int_{B_a} f(tx)dx = \int_{B_ta} f(x)t^{-n}dx$

b) $\int_{B_a} f(tx)dx = \int_{B_tn_a} f(x)tdx$

c) $\int_{\mathbb{R}^n} f(x+y)dx = \int_{\mathbb{R}^n} f(x)dx$ for some $y \in \mathbb{R}^n$

d) $\int_{\mathbb{R}^n} f(tx)dx = \int_{\mathbb{R}^n} f(x)t^n dx$

30 (a), (c)

31. Consider all sequences $\{f_n\}$ of real valued continuous functions on $[0, \infty)$. Identify which of the following statements are correct.

a) If $\{f_n\}$ converges to f pointwise on $[0,\infty)$, then $\lim\limits_{n\to\infty}\int_0^\infty f_n(x)dx = \int_0^\infty f(x)dx$

b) If $\{f_n\}$ converges to f uniformly on $[0,\infty)$,then $\lim\limits_{n\to\infty}\int_0^\infty f_n(x)dx = \int_0^\infty f(x)dx$

c) If $\{f_n\}$ converges to f uniformly on $[0,\infty)$, then f is continuous on $[0,\infty)$

d) There exists a sequence of continuous functions $\{f_n\}$ on $[0,\infty)$,such that $\{f_n\}$ converges to f uniformly on $[0,\infty)$ but $\lim\limits_{n\to\infty}\int_0^\infty f_n(x)dx \neq \int_0^\infty f(x)dx$

31 (c), (d)

32. Find out which of the following series converge uniformly for $x \in (-\pi,\pi)$

a) $\sum\limits_{n=1}^{\infty}\frac{e^{-n|x|}}{n^3}$ b) $\sum\limits_{n=1}^{\infty}\frac{\sin(xn)}{n^5}$ c) $\sum\limits_{n=1}^{\infty}\left(\frac{x}{n}\right)^n$ d) $\sum\limits_{n=1}^{\infty}\frac{1}{((x+\pi)n)^2}$

32 (a), (b), (c)

33. Which one of the following is not uniformly convergent for all $x \in \mathbb{R}$?

a) $\sum\limits_{n=1}^{\infty}\frac{\cos nx}{n^2}$ b) $\sum\limits_{n=1}^{\infty}\frac{\cos nx}{n^3}$ c) $\sum\limits_{n=1}^{\infty}\frac{\sin nx}{n^2}$ d) $\sum\limits_{n=1}^{\infty}\frac{\sin nx}{\sqrt{n}}$

33 (d)

34. Let $f_n(x) = (-x)^n, x \in [0,1]$. Then decide which of the following are true.

a) There exist a pointwise convergent sub sequence of f_n

b) f_n has no pointwise convergent sub sequence

c) f_n converges pointwise everywhere.

d) f_n has exactly one pointwise convergent sub sequence

34 (a)

35. The series $\sum\limits_{n=0}^{\infty}2^{-n}\sin(2^n x)$

a) converges pointwise on $\mathbb{R}$ but not uniformly

b) converges uniformly on $\mathbb{R}$

c) converges uniformly on $[0,\frac{\pi}{2}]$ but not on $\mathbb{R}$

d) does not converge pointwise on $\mathbb{R}$

35 (b)

36. Let $f_n(x) = \sqrt{x^2+n^{-2}}$; $\forall x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then

a) $\langle f_n(x)\rangle$ converges to x uniformly only on a finite interval of $\mathbb{R}$

b) $\langle f_n(x)\rangle$ converges pointwise to $|x|$ on $\mathbb{R}$ but not uniformly there

c) $\langle f_n(x)\rangle$ converges to $|x|$ uniformly on $\mathbb{R}$

d) $\langle f_n(x)\rangle$ converges pointwise to x on $\mathbb{R}$

36 (b)

37. If a function $f(x)$ be such that $f(x) = \sum_{n=0}^{\infty} \phi_n(x)$, where $\phi_n(x) = (1-x)^n$, $0 \le x \le 1$. Then

a) the series does not converge uniformly on $[0,1]$

b) $f(x)$ is continuous on $[0,1]$

c) the series may or may not converge uniformly on $[0,1]$

d) the series converges uniformly on $0,1]$ to x on $\mathbb{R}$

37 (a)

38. Let $f_n[0,1] \to \mathbb{R}$ be given by $f_n(x) = \dfrac{2x^2}{x^2 + (1-2nx)^2}$; $n \in \mathbb{N}$. Then the sequence $\{f_n\}$

a) converges uniformly on $[0,1]$.

b) does not converge uniformly on $[0,1]$ but has a subsequence that converges uniformly on $[0,1]$

c) does not converge pointwise on $[0,1]$

d) converges pointwise $[0,1]$ but does not has a sequence that converges uniformly on $[0,1]$

38 (d)

39. Let $f_n(x) = \dfrac{x}{\{(n-1)x+1\}(nx+1)}$ and $s_n(x) = \sum_{j=1}^{n} f_j(x)$ for $x \in [0,1]$. Then the sequence $\{s_n\}$

a) converges uniformly on $[0,1]$.

b) converges pointwise on $[0,1]$ but not uniformly

c) converges pointwise for $x = 0$ but not for $x \in [0,1]$

d) does not converge for $x \in [0,1]$.

39 (b)

40. Let f be uniformly continuous on $\mathbb{R}$ and $\langle a_n\rangle$ converges to a in $\mathbb{R}$. Let $f_n(x) = f(x + a_n)$ for all $x \in \mathbb{R}$. Then

a) $\langle f_n(x)\rangle$ is only pointwise convergent

b) $\langle f_n(x)\rangle$ is uniformly convergent in $\mathbb{R}$

c) $\langle f_n(x)\rangle$ is divergent sequence

d) $\langle f_n(x)\rangle$ is only bounded but not pointwise convergent

40 (b)

41. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-zero function such that $|f(x)| \leq \dfrac{1}{1+2x^2}$ for all $x \in \mathbb{R}$. Defiune real valued functions f_n on $\mathbb{R}$ for all $n \in \mathbb{N}$ by $f_n(x) = f(x+n)$. Then the series $\sum\limits_{n=1}^{\infty} f_n(x)$ converges uniformly

   a) on $[0,1]$ but not on $[-1,0]$
   b) on $[-1,0]$ but not on $[0,1]$
   c) on both $[0,1]$ and $[-1,0]$
   d) neither on $[0,1]$ nor on $[-1,0]$

41 (c)

42. Let $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}; n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then as $\lim\limits_{n\to\infty} f_n(x) =,$

   a) 0
   b) a continuous function
   c) a bounded function
   d) does not exist

42 (d)

43. Let $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}; n \in \mathbb{N}$ and $x \in [-1,1]$. Then as $n \to \infty$,

   a) $\langle f_n \rangle$ does not converge uniformly in $[-1,1]$
   b) $\lim\limits_{n\to\infty} \displaystyle\int_{-1}^{n} f_n(x)dx \neq 0$
   c) $\langle f'_n(x) \rangle$ does not converge uniformly in $[-1,1]$
   d) $f_n(x), n \in \mathbb{N}$ is not uniformly continuous in $[-1,1]$

43 (c)

44. The series $\sum\limits_{n=1}^{\infty} \dfrac{\sin nx}{\sqrt{n}}; n \in \mathbb{N}$ converges uniformly on

   a) $[5, 2\pi - 5]$
   b) $[10, 2\pi - 10]$
   c) $\left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$
   d) does not converge uniformly

44 (c)

45. The series $\sum\limits_{n=1}^{\infty} \dfrac{\cos nx}{n}; n \in \mathbb{N}$ converges uniformly on

   a) $[10, 2\pi - 10]$
   b) $[4, 2\pi - 4]$
   c) $\left[\dfrac{\pi}{4}, \dfrac{7\pi}{4}\right]$
   d) none of these

45 (c)

46. Let $f_n(x) = \dfrac{1}{1+n^2x^2}$ for $n \in \mathbb{N}, x \in \mathbb{R}$. Which of the following are true ?

   a) f_n converges uniformly on $[0,1]$
   b) f_n converges pointwise on $[0,1]$ to a continuous function
   c) f_n converges uniformly on $[\frac{1}{2}, 1]$
   d) $\lim\limits_{n\to\infty} \displaystyle\int_0^1 f_n(x)dx = \int_0^1 (\lim_{n\to\infty} f_n(x))dx$

46 (c), (d)

47. Let $C_c(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}|$ f is continuous and there exists a compact set K such that $f(x) = 0$ for all $x \in K^c\}$. Let $g(x) = e^{-x^2}$ for all $x \in \mathbb{R}$. For which of the following statements are true?

a) There exists a sequence $\{f_n\}$ in $C_c(\mathbb{R})$ such that $f_n \to g$ uniformly.

b) There exists a sequence $\{f_n\}$ in $C_c(\mathbb{R})$ such that $f_n \to g$ pointwise

c) If a sequence in $C_c(\mathbb{R})$ converges pointwise to g then it must converge uniformly to g

d) There does not exist any sequence in $C_c(\mathbb{R})$ converging pointwise to g.

47 (a), (b)

48. Which of the following sequence $\langle f_n(x) \rangle$ of functions does not converge uniformly on $[0, 1]$?

a) $\dfrac{e^{-x}}{n}$ b) $(1-x)^n$ c) $\dfrac{x^2+nx}{n}$ d) $\dfrac{\sin(nx+n)}{n}$

48 (b)

49. Let $f_n : [1, 2] \to [0, 1]$ be given by $f_n(x) = (2-x)^n; n \in \mathbb{N}$. Let $f(x) = \lim_{n\to\infty} f_n(x)$, $1 \le x \le 2$. Then which of the following is true?

a) f is continuous function on $[1, 2]$

b) f_n converges uniformly to f on $[1, 2]$

c) $\lim_{n\to\infty} \int_1^2 f_n(x)\, dx = \int_1^2 f(x)\, dx$

d) $\forall a \in (1, 2)$, $\lim_{n\to\infty} f_n'(a) \neq f'(a)$

49 (d)

50. Let $f_n : [-1, 1] \to \mathbb{R}$ be given by $f_n(x) = \dfrac{x^{2n}}{1+x^{2n}}; n \in \mathbb{N}$ and $f(x) = \lim_{n\to\infty} f_n(x)$. Let $f(x) = \lim_{n\to\infty} f_n(x)$, $1 \le x \le 2$. Then

a) f is continuous function on $[-1, 1]$

b) $f_n \to f$ uniformly to on $[-1, 1]$

c) f is integrable on $[-1, 1]$

d) none of these

50 (c)

51. Let $f_n(x) = n^2x^2e^{-nx}; n \in \mathbb{N}$ and $x \ge 0$. Then $\langle f_n \rangle$ converges uniformly on

a) $[0, \infty)$

b) $[a, \infty)$ for $a > 0$

c) convergence is not uniform anywhere

d) each f_n is discontinuous

51 (b)

52. Let $f_n(x) = \log(n^2 + x^2); n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then

a) $\langle f_n \rangle$ converges uniformly on $\mathbb{R}$ b) $\langle f_n' \rangle$ converges uniformly on $\mathbb{R}$
c) $\langle f_n \rangle$ converges pointwise on $\mathbb{R}$ d) $\langle f_n' \rangle$ is not convergent on $\mathbb{R}$

52 (b)

53. The series $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{n^p(1+x^{2n})}$ is uniformly convergent for all $x \in \mathbb{R}$ if p is

a) 0 b) $\frac{1}{2}$ c) $\frac{3}{4}$ d) 2

53 (d)

54. If the series $\sum_{n=1}^{\infty} \frac{\cos nx}{n^p}$ converges uniformly on $\mathbb{R}$. Then a value of p is

a) 1 b) $\frac{2}{3}$ c) $\frac{3}{2}$ d) -1

54 (c)

55. If the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$ converges uniformly on $\mathbb{R}$. Then a value of p is

a) 1 b) $\frac{7}{5}$ c) $\frac{5}{7}$ d) $\frac{1}{2}$

55 (b)

[Short Answer Type Questions]

1. Find the limit function f, for the following sequence $\langle f_n \rangle$ of functions
   (a) $f_n(x) = \frac{x}{1+nx}; \quad 0 \le x < \infty.$ Ans: $f(x) = 0, \forall x \in [0, \infty)$
   (b) $f_n(x) = n^2 x(1-x^2)^n; \quad 0 \le x \le 1.$ Ans: $f(x) = 0$
   (c) $f_n(x) = \frac{nx}{1+n^2x^2}; \quad x \in \mathbb{R}.$ Ans: $f(x) = 0, \forall x \in \mathbb{R}$
   (d) $f_n(x) = \frac{\cos nx}{n}; \quad x \in \mathbb{R}^+.$ Ans: $f(x) = 0, \forall x \in \mathbb{R}^+$
   (e) $f_n(x) = \frac{x^{2n}}{1+x^{2n}}; \quad x \in \mathbb{R}.$ Ans: $f(x) = 0, |x| < 1, \frac{1}{2}, |x| = 1, 1, |x| > 1$
   (f) $f_n(x) = \frac{x^{4n}}{1+x^{4n}}; \quad x \in [-1, 1].$ Ans: $f(x) = 0, |x| < 1, \frac{1}{2}, x = \pm 1$
2. Use the definition to examine uniform convergence of the sequence $\langle f_n(x) \rangle$ on $[0, \infty)$, where $f_n(x) =$

   a) $\frac{x}{x+n}$ b) xe^{-nx} c) $n^2x^2e^{-nx}$

   Ans : a) NUC b) UC c) NUC

3. Discuss the uniform convergence of the following sequence of functions $\langle f_n(x)\rangle$ defined by setting $f_n(x) =$

   a) $\dfrac{x}{1+nx^2}; x \in [0,1]$ b) $\dfrac{nx}{1+n^3x^2}; x \in [0,1]$ c) $xe^{-nx}; x \geq 0$

   d) $\begin{cases} nx; & 0 \leq x \leq \frac{1}{n} \\ 1; & \frac{1}{n} < x \leq 1 \end{cases}$

   Ans : a) UC b) UC c) UC d) NUC

4. Study the uniform convergence on $[0,1]$ of the sequence of functions $\langle f_n(x)\rangle$, defined by setting $f_n(x) =$

   a) $\dfrac{1}{1+(nx-1)^2}$ b) $\dfrac{x^2}{x^2+(nx-1)^2}$ c) $x^n(1-x)$

   d) $nx^n(1-x)$ e) $n^3x^n(1-x)^4$ f) $\dfrac{nx^2}{1+nx}$

   g) $\dfrac{1}{1+x^n}$

5. Study the uniform convergence of $\langle f_n(x)\rangle$ on A and B, defined by setting $f_n(x) =$

   a) $\cos^n x(1-\cos^n x)$; $A = \left[0, \frac{\pi}{2}\right]$, $B = \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

   b) $\cos^n x \sin^{2n} x$; $A = \mathbb{R}$, $B = \left[0, \frac{\pi}{4}\right]$

6. A function f defined by $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{10^n}, x \in \mathbb{R}$. Show that f is continuous for any $x \in \mathbb{R}$.

7. Prove or disprove: $\displaystyle\sum_{n=1}^{\infty} 2^{-n}\cos(3^n x)$ represents an everywhere continuous function.

8. If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{a_n x^n}{1+x^{2n}}$ converges uniformly for all $x \in \mathbb{R}$.

9. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n-1}\sqrt{1+nx}}$ is uniformly convergent throughout the positive x-axis.

10. Discuss the convergence and uniform convergence of the series $\sum_{n=1}^{\infty} u_n(x)$, where $u_n(x)$ is given by

    a) $\dfrac{1}{x^2+n^2}; x \in \mathbb{R}$ b) $\dfrac{1}{n^2x^2}; x \in \mathbb{R}/\{0\}$ c) $\sin\left(\dfrac{x}{n^2}\right); x \in \mathbb{R}$

    d) $\dfrac{1}{x^n+1}; x \in \mathbb{R}/\{0\}$ e) $\dfrac{x^n}{1+x^n}; x \geq 0$ f) $\dfrac{(-1)^n}{n+x}; x \geq 0$

    g) $r^n \sin nx; 0 < r < 1, x \in \mathbb{R}$ h) $\dfrac{1}{n^3+n^4x^2}; x \in \mathbb{R}$

Ans : a) UC b) NUC c) UC on $|x| \le a, a > 0$ d) UC on $[a, \infty], a > 1$ e) NUC f) NUC g) UC h) UC

11. For a function f defined on $[a, b]$ set $f_n(x) = \dfrac{[nf(x)]}{n}; n \in \mathbb{N}$ and $x \in [a, b]$. Show that $\langle f_n(x) \rangle$ converges uniformly to $f(x)$ on $[a, b]$.

12. Examine whether $\displaystyle\sum_{n=p}^{\infty} \left[4^{-n}\sin(3^n \pi x) + \frac{\cos(n^2 x)}{p^n}\right]$ is uniformly convergent on $\mathbb{R}$, where p is a positive integer ≥ 2. **Hints :** If the given series is of the form $\sum_{n=p}^{\infty} u_n(x)$, then $|u_n(x)| \le \dfrac{1}{4^n} + \dfrac{1}{p^n} = M_n$.

13. Find where the following series converge pointwise:

    a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+x^n}; x \ne -1$
    b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{1+x^n}; x \ne -1$
    c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + x^n}{1+3^n x^n}; x \ne -\frac{1}{3}$
    d) $\displaystyle\sum_{n=1}^{\infty} \frac{x^{n-1}}{(1-x^n)(1-x^{n+1})}; x \ne -1, 1$
    e) $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2^{n-1}}}{1-x^{2^n}}; x \ne -1, 1$
    f) $\displaystyle\sum_{n=2}^{\infty} \left(\frac{\ln x}{n}\right)^x$
    g) $\displaystyle\sum_{n=1}^{\infty} x^{\ln n}; x > 0$
    h) $\displaystyle\sum_{n=0}^{\infty} \sin^2(2\pi\sqrt{n^2+x^2})$

14. Study the uniform convergence of the following series on the given set A:

    a) $\displaystyle\sum_{n=2}^{\infty} \left[\frac{\pi}{2} - \tan^{-1}\left(n^2(1+x^2)\right)\right]; A = \mathbb{R}$ **Hints:** $\tan^{-1}\dfrac{1}{n^2(1+x^2)} < \dfrac{1}{n^2(1+x^2)} < \dfrac{1}{n^2} = M_n$
    b) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(1+nx)}{nx^n}; A = [2, \infty)$ **Hints:** $M_n = \dfrac{1}{2^{n-1}}$
    c) $\displaystyle\sum_{n=1}^{\infty} n^2 x^2 e^{-n^2|x|}; A = \mathbb{R}$ **Hints:** $M_n = \dfrac{4}{e^2 n^2}$
    d) $\displaystyle\sum_{n=1}^{\infty} x^2\left(1-x^2\right)^{n-1}; A = [-1, 1]$ **Hints:** $f(x) = \begin{cases} 1; & x \in [-1, 1]/\{0\} \\ 0; & x = 0 \end{cases}$ is not continuous
    e) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n!}}\left(x^n + x^{-n}\right); A = \{x \in \mathbb{R} : \frac{1}{2} \le |x| \le 2\}$ **Hints:** $M_n = \dfrac{n^2}{\sqrt{n!}} 2^{n+1}$
    f) $\displaystyle\sum_{n=1}^{\infty} 2^n \sin\frac{1}{3^n x}; \quad A = (0, \infty)$
    g) $\displaystyle\sum_{n=2}^{\infty} \ln\left(1 + \frac{x^2}{n \ln^2 n}\right); A = (-a, a), a > 0$ **Hints:** $M_n = \dfrac{a^2}{n \ln^2 n}$

Ans : a) UC b) UC c) UC d) NUC e) UC f) NUC g) UC

15. Study the continuity on $[0, \infty)$ of the function f defined by $f(x) = \sum_{n=1}^{\infty} \frac{x}{\{(n-1)x+1\}(nx+1)}$.

Hints: $f(x) = \begin{cases} 1; & x > 1 \\ 0; & x = 0 \end{cases}$ is not continuous

16. Study the continuity of the sum of the following series on the domain of its pointwise convergence:

a) $\sum_{n=0}^{\infty} \frac{x^n}{n!} \sin(nx)$ b) $\sum_{n=0}^{\infty} x^{n^2}$ c) $\sum_{n=1}^{\infty} n2^n x^n$ d) $\sum_{n=1}^{\infty} \ln^n(x+1)$

Ans : a) Converges absolutely on $\mathbb{R}$ and the sum function is also continuous on $\mathbb{R}$ b) Converges absolutely on $(-1, 1)$ and the sum function is also continuous on $(-1, 1)$ c) Converges absolutely on $(-1/2, 1/2)$ and the sum function is also continuous on $(-1/2, 1/2)$ d) Converges absolutely on $(\frac{1}{e} - 1, e - 1)$ and the sum function is also continuous on $(\frac{1}{e} - 1, e - 1)$

17. Show that the series $\sum_{n=1}^{\infty} \frac{x \sin(n^2 x)}{n^2}$ converges pointwise to a continuous function on $\mathbb{R}$.

18. Determine whether the series $\sum_{n=1}^{\infty} \left|x\right|^{\sqrt{n}}$ converges pointwise, and study the continuity of the sum. **Ans :** Converges in $(0, 1)$ and the sum function is also continuous there.

19. Let ϕ be continuous on $[0, 1]$. Then the sequence $\langle f_n \rangle$ defined by $f_n(x) = x^n \phi(x)$ converges uniformly on $[0, 1]$ if and only if $\phi(1) = 0$.

20. Verify that the sequence $\langle f_n(x) \rangle$, where $f_n(x) = n \sin \sqrt{4\pi^2 n^2 + x^2}$ converges uniformly on $[0, a], a > 0$. Does $\langle f_n(x) \rangle$ converge uniformly on $\mathbb{R}$?

21. (a) Suppose that the series $\sum_{n=1}^{\infty} u_n(x); x \in A$, converges uniformly on A and that $s : A \to \mathbb{R}$ is bounded. Prove that the series $\sum_{n=1}^{\infty} s(x) u_n(x)$ converges uniformly on A.

    (b) Show by example that boundedness of s is essential. Under what assumption concerning s does the uniform convergence of the series $\sum_{n=1}^{\infty} s(x) u_n(x)$ imply the uniform convergence of $\sum_{n=1}^{\infty} u_n(x)$ on A?

22. Assume that $\langle f_n(x) \rangle$ is a sequence of functions defined on A and such that

a) $f_n(x) \geq 0$ for $x \in A$ and $n \in \mathbb{N}$ b) $f_n(x) \geq f_{n+1}(x)$ for $x \in A$ and $n \in \mathbb{N}$

c) $\sup_{x \in A} f_n(x) \longrightarrow 0$ as $n \to \infty$.

Prove that $\sum_{n=1}^{\infty}(-1)^{n+1}f_n(x)$ converges uniformly on A. **Hints :** For $x \in A$, the series $\sum_{n=1}^{\infty}(-1)^n f_n(x)$ converges by the Leibnitz Theorem. Moreover,

$$\sup_{x\in A}|R_n(x)| = \sup_{x\in A}\left|\sum_{k=n+1}^{\infty}(-1)^{k+1}f_k(x)\right| \leq \sup_{x\in A} f_{n+1}(x)$$

23. Prove that the following series converge uniformly on $\mathbb{R}$:

a) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+x^2}$ b) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt[3]{n+x^2}+x^2}$ c) $\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}+\cos x}$

Hints : Use the Exercise 22.

24. Determine the domain A of pointwise convergence and the domain B of absolute convergence of the series given below. Moreover, study the uniform convergence on the indicated set C.

a) $\sum_{n=1}^{\infty}\frac{1}{n}2^n(3x-1)^n; C=\left[\frac{1}{6},\frac{1}{3}\right]$ b) $\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x+1}{x}\right)^n, C=[-2,-1]$

Ans: a) $A = \left[\frac{1}{6},\frac{1}{2}\right)$ and $B = \left(\frac{1}{6},\frac{1}{2}\right)$ UC on C b) $A = (-\infty,-\frac{1}{2}]$ and $B = (-\infty,-\frac{1}{2})$, UC on C

25. Assume that the functions $f_n, g_n : A \to \mathbb{R}$, $n \in \mathbb{N}$, satisfy the following conditions:

(a) the series $\sum_{n=1}^{\infty}\left|f_{n+1}(x)-f_n(x)\right|$ is uniformly convergent on A

(b) $\sup_{x\in A}|f_n(x)| \to 0$ as $n\to\infty$

(c) the sequence $\langle G_n(x)\rangle$, where $G_n(x)=\sum_{k=1}^{n}g_k(x)$, is uniformly bonded on A.

Prove that the series $\sum_{n=1}^{\infty}f_n(x)g_n(x)$ is uniformly convergent on A.

26. Show that the following series converge uniformly on the indicated set A:

a) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}; A=[0,1]$ b) $\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}; A=[\alpha,2\pi-\alpha], 0<\alpha<\pi$

c) $\sum_{n=1}^{\infty}\frac{\sin(n^2x)\sin(nx)}{n+x^2}; A=\mathbb{R}$ d) $\sum_{n=1}^{\infty}\frac{\sin(nx)\tan^{-1}(nx)}{n}; A=[\alpha,2\pi-\alpha], 0<\alpha<\pi$

e) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^x}; A=[a,\infty), a>0$ f) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^{-nx}}{\sqrt{n+x^2}}; A=[0,\infty)$

27. Let $f, f_n : [0,1] \to \mathbb{R}$ be continuous functions. Compute the following sentence such that both statements (a) and (b) below are true: Let $f_n \to f$:

a) $\lim_{n\to\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)\,dx$ b) $\lim_{n\to\to\infty} \lim_{x\to 0} f_n(x) = \lim_{x\to 0} \lim_{n\to\to\infty} f_n(x)$

Ans : 27 Uniformly on $[0,1]$

28. Let $f_n(x) = x^n$ for $n \in \mathbb{N}$. Which of the following statements are ture?

a) The sequence $\langle f_n(x)\rangle$ converges uniformly on $[\frac{1}{4}, \frac{1}{2}]$

b) The sequence $\langle f_n(x)\rangle$ converges uniformly on $[0,1]$

c) The sequence $\langle f_n(x)\rangle$ converges uniformly on $(0,1)$

Ans : 28 (a)

29. Give examples to illustrate that all the hypotheses in Dini's Theorem (Theorem 5) are essential.

30. Let $\langle r_n\rangle$ be a sequence consisting of all the rational numbers and for $n = 1, 2, \cdots$, we define the functions f_n on $\mathbb{R}$ by $f_n(x) = \begin{cases} 1; & x = r_n \\ 0; & \text{otherwise} \end{cases}$. Prove that $\langle f_n(x)\rangle$ converges pointwise but not uniformly on every interval of $\mathbb{R}$.

[Long Answer Type Questions]

1. Determine whether the sequence $\langle f_n(x)\rangle$ converges uniformly on A, defined by setting $f_n(x) =$

a) $\tan^{-1}\frac{2x}{x^2+n^3}; A = \mathbb{R}$

b) $n\ln\left(1+\frac{x^2}{n}\right); A = \mathbb{R}$

c) $n\ln\frac{1+nx}{nx}; A = (0,\infty)$

d) $\sqrt[2n]{1+x^{2n}}; A = \mathbb{R}$

e) $\sqrt[n]{2^n + |x|^n}; A = \mathbb{R}$

f) $\sqrt{n+1}\sin^n x\cos x; A = \mathbb{R}$

g) $n(\sqrt[n]{x}-1); A = [1,a], a > 1$

2. Prove that for the sequence $\langle f_n\rangle$, $f_n \to f$ pointwise on a point set $E \subset \mathbb{R}$, the convergence is uniform.

3. (a) Suppose that $\langle f_n\rangle \in \mathcal{F}(E;\mathbb{R})^{\mathbb{N}}$ converges to f uniformly on E and that each f_n is bounded on E. Show that $\langle fn\rangle$ is uniformly bounded; i.e., there is an $M > 0$ such that $|f_n(x)| \le M$ for all $n \in \mathbb{N}$ and all $x \in E$.

(b) Let $f_n(x) = \frac{1}{x} + \frac{x}{n}$ for all $x \in (0,1]$. Show that $\langle f_n\rangle$ converges uniformly to $f(x) = x^{-1}$ on $(0,1]$ and yet the f_n and f are all unbounded on $(0,1]$.

4. Let $E \subset \mathbb{R}$ and $\langle f_n\rangle \in \mathcal{F}(E;\mathbb{R})^{\mathbb{N}}$. Prove that, a sequence of functions $\langle f_n(x)\rangle_n$ defined on E_0 converges uniformly on $E_0 \subset E$ if and only if corresponding to an $\epsilon > 0$, $\exists N_0(\epsilon) \in \mathbb{N}$ such that $\left|f_{n+p}(x) - f_n(x)\right| < \epsilon$, for $n \ge N$ and $\forall x \in E_0$.

5. Prove that, a sequence $\langle f_n \rangle \in \mathcal{F}(E;\mathbb{R})^{\mathbb{N}}$, where $E \subset \mathbb{R}$; converges uniformly on $E_0 \subset E$ if and only if $\lim\limits_{n \longrightarrow \infty} \sup \Big\{ |f_n(x) - f(x)| : x \in E_0 \Big\} = 0$.

6. Prove that, a sequence $\langle f_n \rangle \in \mathcal{F}(E;\mathbb{R})^{\mathbb{N}}$, where $E \subset \mathbb{R}$; converges uniformly on $E_0 \subset E$ if and only if, corresponding to an $\varepsilon > 0$, $\exists N = N(\varepsilon) \in \mathbb{N}$, depends on ε only, such that

$$\sup \Big\{ |f_m(x) - f_n(x)| : x \in E_0 \Big\} < \varepsilon; \quad \text{whenever } m, n \geq N$$

7. Find the convergence set of each of the following sequences $\langle f_n(x) \rangle$ of functions on $[0,1]$

a) $f_n(x) = n^2 x^n (1-x)$ b) $f_n(x) = \left(1 + \frac{x}{n}\right)^n$ c) $f_n(x) = nx(1-x)^n$.

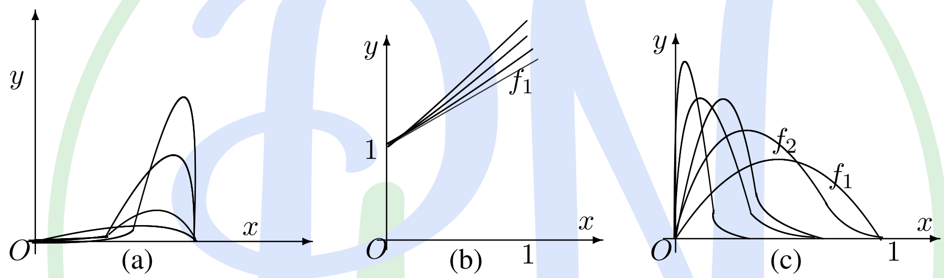

Figure 16: Figures of $f_n(x)$

as depicted in the Figures 16 (a), (b) and (c) respectively. Also find sets on which these converge uniformly.

8. If a sequence of functions $\langle f_n \rangle$ converges uniformly to f on $[a,b]$ and g is a bounded function on $[a,b]$, show that the sequence of functions $\langle gf_n \rangle$ converges uniformly to gf on $[a,b]$.

9. If a sequence of functions $\langle f_n \rangle$ converges uniformly on $[a,b]$ to a function f and if $c \in [a,b]$ such that $\lim\limits_{n \to c} f_n(x) = \alpha_n$, $n \in \mathbb{N}$, show that the sequence $\langle \alpha_n \rangle$ is convergent.

10. If $\langle f_n \rangle$ be a sequence of continuous real valued functions converging uniformly to f on a set $E(\subset \mathbb{R})$. Show that $\lim\limits_{n \to \infty} f_n(x_n) = f(x)$ for any sequence $\langle \alpha_n \rangle$ in E converging to a point x in E.

11. Suppose K is a compact set and let $\langle f_n(x) \rangle$ be a sequence of continuous functions converging pointwise to a continuous functions f and $f_n \geq f_{n+1}$, then $f_n \to f$ uniformly. **Hints :** Theorem 5

12. (Dini's Theorem of uniform convergence of a series of functions) Let $I \subset \mathbb{R}$ be a compact interval and suppose that $\langle u_n \rangle \in \mathcal{F}(E;\mathbb{R})^{\mathbb{N}}$ is a sequence of continuous functions such that the series $\sum\limits_{n=1}^{\infty} u_n$ converging pointwise to a continuous function $s : I \to \mathbb{R}$. If the f_n is increasing (i.e., $u_n(x) \leq u_{n+1}(x)$ for all $x \in I$ and $n \in \mathbb{N}$) or decreasing (i.e., $u_n(x) \geq u_{n+1}(x)$ for all $x \in I$ and $n \in \mathbb{N}$), then prove that $\sum\limits_{n=1}^{\infty} u_n$ converges to s uniformly on I.

13. For each $n \geq 1$, let $f_n(x)$ be a monotonic increasing real valued function on $[0,1]$ such that the sequence $\langle f_n(x) \rangle$ converges pointwise to a function $f \equiv 0$. Pick out the true statements from the following:

    a) Sequence $\langle f_n(x) \rangle$ converges to f uniformly

    b) If the functions f_n are non-negative, then f_n must be continuous for sufficiently large n **Hints :** $f_n(x) = \begin{cases} 0; & x \in [0,1) \\ \frac{1}{n}; & x = 1 \end{cases}$

    Ans: 13 (a)

14. Let f_n and f be continuous functions on an interval $[a,b]$ and assume that $f_n \to f$ uniformly on $[a,b]$. Pick out the true statements:

    a) If f_n is Riemann-integrable, then f is Riemann-integrable.

    b) If f is continuously differentiable, then f is continuously differentiable.**Hints :** $f_n(x) = \sqrt{x^2 + \dfrac{1}{n^2}}$ in $[-1,1]$

    c) If $x_n \to x$ in $[a,b]$, then $f_n(x) \to f(x)$.

    Ans: 14 (a), (c)

15. Pick out the sequence $\langle f_n \rangle$ which are uniformly convergent **NBHM'09**

    a) $f_n(x) = nxe^{-nx}$ on $(0,\infty)$ b) $f_n(x) = x^n$ on $[0,1]$ c) $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$ on $\mathbb{R}$

    Ans: 15 (c)

16. Test the following for uniformly convergent

    a) The sequence of functions $\left\langle \dfrac{x^n}{1+x^n} \right\rangle$ over $[0,2]$ b) The series $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{n^2+1}$ over $[0,2]$

    c) The sequence of functions $\left\langle n^2x^2e^{-nx} \right\rangle$ over $(0,1)$

    Ans: 16 (b)

17. In each of the following cases, examine whether the given sequence (or series) of functions converges uniformly over the given domain NBHM'11

    a) $f_n(x) = \dfrac{nx}{1+nx}; x \in (0,1)$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{n \sin nx}{e^n}; x \in [0,\pi]$ c) $f_n(x) = \dfrac{x^n}{1+x^n}; x \in [0,2]$

    Ans: 17 (b)

18. Which of the following sequence (or series) of functions are uniformly convergent on $[0,1]$? NBHM'13

a) $f_n(x) = \Big(\cos(\pi n! x)\Big)^n$ b) $f(x) = \sum_{n=1}^{\infty} \frac{\cos(n^6 x)}{n^3}$ c) $f_n(x) = n^2 x(1-x^2)^n$

Ans: 18 (b)

19. Let $g_n(x) = \frac{1}{n}\Big[f\Big(x + \frac{1}{n} - f(x)\Big]$, where $f : \mathbb{R} \to \mathbb{R}$ a continuous function. Which of the following statements are ture? NBHM'14

a) If $f(x) = x^3$, then $g_n \to f'$ uniformly on $\mathbb{R}$ as $n \to \infty$

b) If $f(x) = x^2$, then $g_n \to f'$ uniformly on $\mathbb{R}$ as $n \to \infty$

c) If f is differentiable and f' is uniformly continuous on $\mathbb{R}$, then $g_n \to f'$ uniformly on $\mathbb{R}$ as $n \to \infty$

Ans: 19 (b)

20. Which of the following statements are true ? NBHM'14

a) The series $\sum_{n=1}^{\infty} \frac{x^2}{1+n^2x^2}$ does not converge uniformly on $\mathbb{R}$

b) The series $\sum_{n=1}^{\infty} \frac{x^2}{1+n^2x^2}$ converges uniformly on $\mathbb{R}$

c) The sum of the series $\sum_{n=1}^{\infty} \frac{\sin nx^2}{1+n^2}$ defines a continuously differentiable function on $\mathbb{R}$

Ans: 20 (b), (c)

21. Which of the following statements are true ? NBHM'17

a) The series $\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}$ is UC on $[-1,1]$ b) $\lim_{n\to\infty} \int_{\pi/2}^{\pi} \frac{\sin nx}{nx^5} = \pi$

c) Define $x \in \mathbb{R}$, $f(x) = \sum_{n=0}^{\infty} \frac{\sin nx^2}{1+n^3}$, then f is a continuously differentiable function.

Ans: 21 (c)

22. Let $\langle f_n(x) \rangle$ be a sequence of continuous functions defined on $[0,1]$. Assume that $f_n(x) \to f(x)$ for each $x \in [0,1]$. Which of the following conditions imply that this convergence is uniform ? NBHM'18

a) The function f is continuous

b) $f_n(x) \downarrow f(x)$ for every $x \in [0,1]$ **Hints :** $f_n(x) = 1 - x^{1/n}$

c) The function f is continuous and $f_n(x) \downarrow f(x)$ for every $x \in [0,1]$ **Hints :** Dini's Theorem 5

Ans: 22 (c)

23. Which of the following statements are true ? NBHM'19

a) The sequence of functions $\langle f_n(x)\rangle$, defined by $f_n(x) = x^n(1-x)$, is uniformly convergent on $[0,1]$

b) The sequence of functions $\langle f_n(x)\rangle$, defined by $f_n(x) = n\log\left(1+\frac{x^2}{n}\right)$, is uniformly convergent on $\mathbb{R}$

c) The series $\sum_{n=1}^{\infty} 2^n \sin\frac{1}{3^n x}$ is uniformly convergent on $[1,\infty)$

Ans: 23 (a), (c)

24. Test the uniform convergence of the sequence of functions $\langle f_n\rangle$, defined by $f_n : [-k,k] \to \mathbb{R}$, where, $f_n(x) = \frac{\log(1+n^2x^2)}{n}; x\in[-k,k], k>0.$

25. Show that the sequence $f_n : x \to x^n$ converges for each $x \in I = \{x : 0 \le x \le 1\}$ but that the convergence is not uniform.

26. In each of the following problems, show that the sequence $\langle f_n\rangle$ converges to f for each $x \in I$ and determine whether or not the convergence is uniform:

(a) $f_n : x \longrightarrow \frac{2x}{1+nx}; f(x) \equiv 0; I = \{x : 0 \le x \le 1\}$

(b) $f_n : x \longrightarrow \frac{\cos nx}{\sqrt{n}}; f(x) \equiv 0; I = \{x : 0 \le x \le 1\}$

(c) $f_n : x \longrightarrow \frac{n^3x}{1+n^4x}; f(x) \equiv 0; I = \{x : 0 \le x \le 1\}$

(d) $f_n : x \longrightarrow \frac{n^3x}{1+n^4x^2}; f(x) \equiv 0; I = \{x : a \le x < \infty, a > 0\}$

(e) $f_n : x \longrightarrow \frac{nx^2}{1+nx}; f(x) \equiv x; I = \{x : 0 \le x \le 1\}$

(f) $f_n : x \longrightarrow \frac{1}{\sqrt{x}} + \frac{1}{n}\cos\left(\frac{x}{n}\right); f(x) \equiv \frac{1}{\sqrt{x}}; I = \{x : 0 < x \le 2\}$

(g) $f_n : x \longrightarrow \frac{\sin nx}{2nx}; f(x) \equiv 0; I = \{x : 0 < x < \infty\}$

(h) $f_n : x \longrightarrow x^n(1-x)\sqrt{n}; f(x) \equiv 0; I = \{x : 0 \le x \le 1\}$

(i) $f_n : x \longrightarrow \frac{1-x^n}{1-x}; f(x) \equiv \frac{1}{1-x}; I = \{x : -\frac{1}{2} \le x \le \frac{1}{2}\}$

(j) $f_n : x \longrightarrow nxe^{-nx^2}; f(x) \equiv 0; I = \{x : 0 \le x \le 1\}$

27. (a) $\sum_{n=1}^{\infty}(xe^{-x})^n; x \in [0,2]$

(b) $\sum \frac{sin(x^2+n^2x)}{n(n+1)}$

(c) $\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n}}{n^p(1+x^{2n})}$

28. Show that the series $x^4+\dfrac{x^4}{1+x^4}+\dfrac{x^4}{(1+x^4)^2}+\dfrac{x^4}{(1+x^4)^3}+\cdots$ is not uniformly convergent on $[0,1]$.

29. Prove that if a series of continuous functions converges uniformly then the sum function is also continuous.

30. Formulate and prove a result about the derivative of the sum of a convergent series of differentiable functions.

31. Let $\langle u_n\rangle$ be a sequence of R-integrable functions on a compact interval $[a,b]\subset\mathbb{R}$. If the infinite series $\sum_{n=1}^{\infty} u_n$ converges uniformly to sum s on $[a,b]$, then

    (a) $s\in\mathcal{R}[a,b]$, i.e., s is R-integrable on $[a,b]$, and

    (b) $\displaystyle\int_a^x s(t)dt=\int_a^x\Big[\sum_{n=1}^{\infty}u_n(t)\Big]dt=\sum_{n=1}^{\infty}\Big[\int_a^x u_n(t)dt\Big]$

32. Let $\langle f_n(x)\rangle$ be a sequence of functions in $\mathcal{C}^1[0,1]$ such that $f_n(0)=0$ for all $n\in\mathbb{N}$. Which of the following statements are true ? NBHM'19

    a) If the sequence $\langle f_n(x)\rangle$ converges uniformly on $[0,1]$, then the limit function is in $\mathcal{C}^1[0,1]$

    b) If the sequence $\langle f_n'(x)\rangle$ is uniformly convergent over $[0,1]$, then the sequence $\langle f_n(x)\rangle$ is also uniformly convergent over the same interval.

    c) If the sequence $\langle f_n(x)\rangle$ converges uniformly on $[0,1]$, then the limit function is in $\mathcal{C}^1[0,1]$

    d) If the series $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly on $[0,1]$ to a function g, then g is Riemann integrable and $\displaystyle\int_0^1 g(t)dt=\sum_{n=1}^{\infty} f_n(x)$

    Ans: 32 (b), (c)

33. Let $\{r_1,r_2,\cdots,r_n,\cdots\}$ be an enumeration of the rationals in the interval $[0,1]$. Define, for $n\in\mathbb{N}$, and for each $x\in[0,1]$ $f_n(x)=\begin{cases}1; & \text{if } x\in\{r_1,r_2,\cdots\}\\ 0; & \text{otherwise}\end{cases}$ Which of the following statements are true? NBHM'19

    a) The function f_n is Riemann integrable over $[0,1]$ for each $n\in\mathbb{N}$

    b) The sequence $\langle f_n(x)\rangle$ is pointwise convergent and the limit function is Riemann integrable over the interval $[0,1]$ **Hints :** $f(x)=\begin{cases}1; & \text{if } x\in\mathbb{Q}\\ 0; & \text{otherwise}\end{cases}$

    c) The sequence $\langle f_n(x)\rangle$ is pointwise convergent but the limit function is not Riemann integrable over $[0,1]$

Ans: 33 (c)

34. Let $\langle f_n \rangle$ be a sequence of continuous real valued functions defined on $\mathbb{R}$ converging uniformly on $\mathbb{R}$ to a function f. Which of the following statements are true? NBHM'16

    a) If each of the functions f_n is bounded, then f is also bounded.

    b) If each of the functions f_n is uniformly continuous, then f is also uniformly continuous.

    c) If each of the functions f_n is integrable, then $\lim\limits_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\,dx = \int_{-\infty}^{\infty} f(x)dx$.

Ans: 34 (a),(b)

35. Let $\langle f_n(x) \rangle$ be a sequence of non-negative continuous functions defined on $[0,1]$. Assume that $f_n(x) \to f(x)$ for each $x \in [0,1]$. Which of the following conditions imply that $\lim\limits_{n\to\infty}\int_{)}^{1} f_n(x)dx = \int_0^1 f(x)dx$? NBHM'18

    a) $f_n(x) \uparrow f(x)$ for every $x \in [0,1]$

    b) $f_n(x) \leq f(x)$ for every $x \in [0,1]$. **Hints :** Actually the limit of non-negative continuous functions may not be R integrable function but the additional condition $f_n(x) \leq f(x)$ makes not only f R integrable but the limit of integration is also convergent.

    c) The function f is continuous **Hints :** Consider the functions $f_n : [0,1] \to \mathbb{R}$, defined by $f_n(x) = \begin{cases} n - n^2x; & \text{if } 0 < x < \frac{1}{n} \\ 0; & otherwise \end{cases}$

Ans: 35 (a), (b)

36. Let $\langle f_n \rangle$ be a sequence of bounded real valued functions on $[0,1]$ converging to f at all points of this interval. Which of the following statements are ture? NBHM'14

    a) If f_n and f are all continuous, then $\lim\limits_{n\to\infty}\int_0^1 f_n(x)dx = \int_0^1 f(x)dx$ **Hints :** Example 10

    b) If $f_n \to f$ uniformly, on $[0,1]$, then $\lim\limits_{n\to\infty}\int_0^1 f_n(x)dx = \int_0^1 f(x)dx$. **Hints :** Theorem 14

    c) If $\int_0^1 \Big|f_n(x) - f(x)\Big|dx \to 0$ and $n \to\to \infty$, then $\lim\limits_{n\to\infty}\int_0^1 f_n(x)dx = \int_0^1 f(x)dx$ **Hints :** Using the inequality $\Big|\lim\limits_{n\to\infty}\int_0^1 f_n(x)dx - \int_0^1 f(x)dx\Big| \leq \lim\limits_{n\to\infty}\int_0^1 |f_n(x) - f(x)|\,dx$

Ans: 36 (b),(c)

37. Let $\langle f_n(x)\rangle$ and f be integrable functions on $[0,1]$ such that $\lim_{n\to\infty}\int_0^1 |f_n(x)-f(x)|dx = 0$. Which of the following statements are true? NBHM'19

a) $f_n(x) \to f(x)$, as $n\to\infty$, for almost every $x\in[0,1]$

b) $\lim_{n\to\infty}\int_0^1 f_n(x)dx = \int_0^1 f(x)dx$

c) If $\langle g_n(x)\rangle$ is a uniformly bounded sequence of continuous functions converging pointwise to a function g, then $\int_0^1 |f_n(x)g_n(x) - f(x)g(x)|dx = 0$ as $n\to\infty$

Ans: 37 (b), (c)

38. Let $\langle f_n\rangle$ be a sequences of continuous real valued functions defined on $[a,b]$ and also let $\langle a_n\rangle$ and $\langle b_n\rangle$ be two sequences on $[a,b]$ such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. If $\langle f_n\rangle$ converges uniformly to f on $[a,b]$, then show that $\lim_{n\to\infty}\int_{a_n}^{b_n} f_n(x)dx = \int_a^b f(x)dx$.

39. Let $f(x) = \begin{cases} a_{n+1}; & \text{if } x = n\in[0,2014]\cap\mathbb{Z} \\ 0; & \text{otherwise}\end{cases}$, where $\langle a_n\rangle$ is a real sequence. Is f integrable on $[0,2014]$? If so, find $\int_0^{2014} f(x)dx$. Ans: 0

40. Let $\langle f_n\rangle$ be a sequence defined by $f_n(x) = \begin{cases} n^2x; & 0\le x\le \frac{1}{n} \\ 2n - n^2x; & \frac{1}{n} < x \le \frac{2}{n} \text{ for } \forall n\ge 2 \\ 0; & \frac{2}{n} < x \le 1\end{cases}$. Then show that

(a) The sequence $\langle f_n\rangle$ is not uniformly convergent on $[0,1]$.

(b) each f_n is Riemann integrable on $[0,1]$

(c) $\langle f_n\rangle$ has a pointwise limit f which is also Riemann integrable on $[0,1]$ and

(d) $\lim_{n\to\infty}\int_0^1 f_n(x)dx \neq \int_0^1 f(x)dx$.

41. For what values of p, the sequence $\langle f_n\rangle$ defined on $[0,1]$ by $f_n(x) = \dfrac{nx}{1+n^2x^p}(p>0)$ for $x\in[0,1]$ converges uniformly on $[0,1]$. Examine further for $p=2$ and $p=4$ if $\lim_{n\to\infty}\int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx$.

42. Show that the sequence of functions $\langle f_n\rangle$ defined on $[0,1]$ by $f_n(x) = n^2x(1-x^2)^n, n\in\mathbb{N}$ for $x\in[0,1]$ converges pointwise to a function f on $[0,1]$. By establishing $\lim_{n\to\infty}\int_0^1 f_n(x)dx \neq \int_0^1 f(x)dx$, show that the sequence $\langle f_n\rangle$ is not uniformly convergent on $[0,1]$.

43. Show that for the sequence of continuous functions $\langle f_n \rangle$, where $f_n(x) = \begin{cases} 1 - nx; & x \in [0, \frac{1}{n}] \\ 0; & x \in (\frac{1}{n}, 1] \end{cases}$ the pointwise limit function is not continuous in $[0,1]$ and hence deduce that $\lim\limits_{x\to 0}\{\lim\limits_{n\to\infty} f_n(x)\} \neq \lim\limits_{n\to\infty}\{\lim\limits_{x\to 0} f_n(x)\}$.

44. Show that the sequence $\langle f_n \rangle$ of functions defined by $f_n(x) = \int_0^x \frac{t\,dt}{1+n^2 t}, x \geq 0$ converges uniformly to 0 on $[0, a)$, $a > 0$; but not on $[0, \infty)$.

a)

5 The Weierstrass Approximation Theorem

The name Weierstrass has occurred frequently in this chapter. In fact Karl Weierstrass (1815-1897) revolutionized analysis with his examples and theorems. This section is devoted to one of his most striking results. We introduce it with a motivating discussion.

Definition 5. Bernstein polynomial : For $f : [0,1] \longrightarrow \mathbb{R}$, let $\mathcal{B}_n(f,x)$ be the Bernstein polynomial of order n associated with the function f, defined by

$$\mathcal{B}_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} f\Big(\frac{k}{n}\Big) x^k (1-x)^{n-k}. \tag{6}$$

The Bernstein polynomial of the continuous function $f_0(x) = 1$ is given by

$$\begin{aligned} \mathcal{B}_n(f_0) &= \sum_{k=0}^{n} \binom{n}{k} f_0\Big(\frac{k}{n}\Big) x^k (1-x)^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = \Big(x + (1-x)\Big)^n = 1 \end{aligned} \tag{7}$$

The Bernstein polynomial of the continuous function $f_1(x) = x$ is given by

$$\begin{aligned} \mathcal{B}_n(f_1) &= \sum_{k=0}^{n} \binom{n}{k} f_1\Big(\frac{k}{n}\Big) x^k (1-x)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \frac{k}{n} \cdot x^k (1-x)^{n-k} \\ &= x \sum_{k=1}^{n} \binom{n-1}{k-1} x^{k-1} (1-x)^{n-k} = x\Big(x + (1-x)\Big)^{n-1} = x = f_1(x) \end{aligned} \tag{8}$$

The Bernstein polynomial of the continuous function $f_2(x) = x^2$ is given by

$$\begin{aligned}
\mathcal{B}_n(f_2) &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n}\right)^2 \cdot x^k(1-x)^{n-k} = \sum_{k=0}^{n} \frac{k}{n} \cdot \binom{n-1}{k-1} \cdot x^k(1-x)^{n-k} \\
&= \sum_{k=0}^{n} \left\{\frac{k-1}{n-1} \cdot \frac{n-1}{n} + \frac{1}{n}\right\} \cdot \binom{n-1}{k-1} \cdot x^k(1-x)^{n-k} \\
&= \sum_{k=0}^{n} \left[\left\{1 - \frac{1}{n}\right\} \cdot \binom{n-2}{k-2} + \frac{1}{n}\binom{n-1}{k-1}\right] \cdot x^k(1-x)^{n-k} \\
&= \left(1 - \frac{1}{n}\right) \sum_{k=2}^{n} \cdot \binom{n-2}{k-2} x^k(1-x)^{n-k} + \frac{1}{n}\sum_{k=1}^{n} \frac{1}{n}\binom{n-1}{k-1} x^k(1-x)^{n-k} \\
&= \left(1 - \frac{1}{n}\right) x^2 + \frac{1}{n} x = \left(1 - \frac{1}{n}\right) f_2(x) + \frac{1}{n} f_1(x) \qquad (9)
\end{aligned}$$

This shows that the Bernstein polynomial $\mathcal{B}_n(f_2)$ converges to $f_2(x)$ on any bounded subset of $\mathbb{R}$.

EXAMPLE **56.** Prove that $\dfrac{x(1-x)}{n} = \sum_{k=0}^{n} x^k(1-x)^{n-k}\left(x - \frac{k}{n}\right)^2$.

Solution: Differentiate both sides of Eq. (7), we obtain

$$\begin{aligned}
0 &= \sum_{k=0}^{n} \binom{n}{k} \left[kx^{k-1}(1-x)^{n-k} + x^k(n-k)(1-x)^{n-k-1} \cdot (-1)\right] \\
&= \sum_{k=0}^{n} \binom{n}{k} (k - nx) x^{k-1}(1-x)^{n-k-1} \qquad (10)
\end{aligned}$$

Multiply both sides of Eq. (10) by $x(1-x)$, we get

$$0 = \sum_{k=0}^{n} \binom{n}{k} (k - nx) x^k (1-x)^{n-k} \qquad (11)$$

Differentiate both sides of Eq. (11), we obtain

$$\begin{aligned}
0 &= -n \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} + \sum_{k=0}^{n} \binom{n}{k} (k - nx)\left\{(k-nx)x^{k-1}(1-x)^{n-k-1}\right\} \\
&= -n + \sum_{k=0}^{n} \binom{n}{k} (k-nx)^2 x^{k-1}(1-x)^{n-k-1} \qquad (12)
\end{aligned}$$

Multiply both sides of Eq. (12) by $x(1-x)$, we get

$$\begin{aligned}
& 0 = -nx(1-x) + \sum_{k=0}^{n} \binom{n}{k} (k-nx)^2 x^k(1-x)^{n-k} \\
\text{or,} \quad & nx(1-x) = \sum_{k=0}^{n} \binom{n}{k} (k-nx)^2 x^k (1-x)^{n-k} \\
\text{or,} \quad & \frac{x(1-x)}{n} = \sum_{k=0}^{n} x^k(1-x)^{n-k}\left(x - \frac{k}{n}\right)^2.
\end{aligned}$$

THEOREM **18.** *For $f : [0,1] \longrightarrow \mathbb{R}$, let $\mathcal{B}_n(f,x)$ be the Bernstein polynomial of order n of the function f as in Eq. (6). If f is continuous on $[0,1]$, then $\langle \mathcal{B}_n(f,x)\rangle$ converges uniformly on $[0,1]$ to f.*

Proof: Using the equality (7), we get

$$f(x) = \sum_{k=0}^{n} f(x)\binom{n}{k}x^k(1-x)^{n-k}.$$

Consequently,

$$\Big|\mathcal{B}_n(f,x) - f(x)\Big| \leq \sum_{k=0}^{n}\Big|f\Big(\frac{k}{n}\Big) - f(x)\Big|\binom{n}{k}x^k(1-x)^{n-k} \tag{13}$$

By the uniform continuity of f on $[0,1]$, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$|f(x) - f(x')| < \varepsilon;\ \text{whenever}\ |x - x'| < \delta; x, x' \in [0,1]$$

Clearly, there is $M > 0$ such that $|f(x)| \leq M$ for $x \in [0,1]$. Then the set $\{0,1,2,\cdots,n\}$ can be decomposed into the two sets : $A = \Big\{k : \Big|\frac{k}{n} - x\Big| < \delta\Big\}$ and $B = \Big\{k : \Big|\frac{k}{n} - x\Big| \geq \delta\Big\}$. If $k \in A$, then $\Big|f\Big(\frac{k}{n}\Big) - f(x)\Big| < \varepsilon$ and so

$$\sum_{x\in A}\Big|f\Big(\frac{k}{n}\Big) - f(x)\Big|\binom{n}{k}x^k(1-x)^{n-k} < \varepsilon\sum_{x\in A}\binom{n}{k}x^k(1-x)^{n-k} \leq \varepsilon \tag{14}$$

If $k \in B$, then $\frac{(k-nx)^2}{n^2\delta^2} \geq 1$, then

$$\sum_{k\in B}\Big|f\Big(\frac{k}{n}\Big) - f(x)\Big|\binom{n}{k}x^k(1-x)^{n-k}$$
$$\leq \frac{2M}{n^2\delta^2}\sum_{k\in B}(k-nx)^2\binom{n}{k}x^k(1-x)^{n-k} \leq \frac{M}{2n\delta^2} \tag{15}$$

This Equation (15) combined with Eqs. (13) and (14) yields

$$\Big|\mathcal{B}_n(f,x) - f(x)\Big| \leq \varepsilon + \frac{M}{2n\delta^2};\ \ x \in [0,1]$$

This proves the theorem. □

THEOREM **19** (Approximation theorem of Weierstrass)**.** *If $f : [a,b] \to \mathbb{R}$ be a continuous function on an interval $[a,b]$, then for $\varepsilon > 0$ there is a polynomial $p(x)$ such that*

$$|f(x) - p(x)| < \epsilon;\ \ x \in [a,b]$$

In particular, there exists a sequence $\langle p_n(x)\rangle$ of polynomial functions such that $p_n(x) \longrightarrow f(x)$ on $[a,b]$.

Proof: We know that for each $n \in \mathbb{N}$, $\exists$ a polynomial $p_n(x)$ such that

$$|p_n(x) - f(x)| < \frac{1}{n}; \;\; \forall x \in [a,b]$$

Taking $n = 1, 2, 3, \cdots$, we get the sequence of polynomials $\langle p_n(x) \rangle$. Let $\varepsilon > 0$ be chosen arbitrary. Choose $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. Then

$$|p_n(x) - f(x)| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon; \;\; \forall n \geq N \text{ and } \forall x \in [a,b]$$

N depends only on ε. Therefore, $\langle p_n(x) \rangle$ converges uniformly to f on $[a,b]$.

The geometrical significance of this this theorem lies in the fact that, the graph (Fig. 17) of the

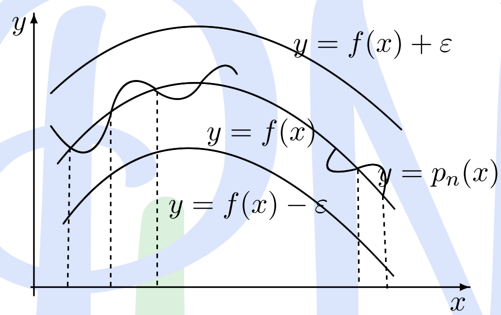

Figure 17: The Weierstrass Approximation Theorem

polynomial $p_n(x)$ is confined within the region bounded by $y = f(x) - \varepsilon$ and $y = f(x) + \varepsilon$ for all $x \in (a,b)$. This theorem does not guarantee the existence of an polynomial, even if how to construct the polynomial.

EXAMPLE **57.** *Let $f \in \mathcal{C}[0,1]$. Determine the cases where the given condition implies that $f \equiv 0$:* **NBHM'07,'10**

a) $\int_0^\pi x^n f(x) dx = 0$ *for all* $n \geq 0$ *b)* $\int_0^\pi x^n f(x) \cos nx dx = 0$ *for all* $n \geq 0$

c) $\int_0^\pi x^n f(x) \sin nx dx = 0$ *for all* $n \geq 1$

Solution: (a) For any given $f \in \mathcal{C}[0,1]$, by Weierstrass polynomial approximation theorem, there exists a function $\phi(x) = \sum_{k=0}^{m} a_k x^k$ such that $|f(x) - \phi(x)| < \varepsilon$. Since $\int_0^\pi x^n f(x) dx = 0$ for all $n \geq 0$, we have

$$\int_0^\pi \phi(x) f(x) dx = \sum_{k=0}^{m} a_k \int_0^\pi x^k f(x) dx = 0$$

Thus, as f is continuous on a compact set, it must be bounded (say bounded by M). Consider

$$\begin{aligned} \int_0^\pi f^2(x)\, dx &= \int_0^\pi f(x)[f(x) - \phi(x)] dx + \int_0^\pi \phi(x) f(x) dx \\ &< \varepsilon M \pi + 0 \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\int_0^{\pi} f^2(x)\, dx = 0$, implies $f \equiv 0$ as it is continuous. So this option is correct.

(b) Extend f to f^e on $[-\pi, \pi]$ so that $f^e(-x) = f(x)$ for $x \in [0, \pi]$. Then, we have

$$\int_{-\pi}^{\pi} f^e(x) \cos nx\, dx = 2 \int_0^{\pi} f(x) \cos nx dx = 0; n \geq 0$$

as f^e and $\cos$ both are even functions. Similarly, we have $\int_{-\pi}^{\pi} f^e(x) \sin nx\, dx = 0$, as $\sin$ is odd, $n \geq 1$. Thus all the Fourier coefficients of f^e are zero. By Parseval's theorem, $\int_{-\pi}^{\pi} \left|f^e(x)\right|^2 dx = 0$. The continuity of f^e then imply $f^e = 0$. Because of $f^e = f$ on $[0, \pi]$, we have $f \equiv 0$. Therefore option (b) is also correct.

(c) This option is again correct, similarly as option (b).

Problem Set

Short answer type questions

1. Let P denote the set of all polynomials in the real variable x which varies over the interval $[0, 1]$. What is the closure of P in $\mathcal{C}[0, 1]$? **Ans :** $\mathcal{C}[0, 1]$

   Hints : Weierstrass approximation theorem.

Long answer type questions

1. Establish the identity: $\dfrac{x(1-x)}{n} = \sum_{k=0}^{n} x^k (1-x)^{n-k} \left(x - \dfrac{k}{n}\right)^2$. **Hints :** Differentiate Eq. (7)

2. Let $f : [0, 1] \longrightarrow \mathbb{R}$ be continuous. Assume that $\int_{)}^{1} x^n f(x) dx = 0$ for $n \in \mathbb{N}$, then prove that $f = 0$.

3. Show that there exists no sequence of polynomials $\langle p_n \rangle$ such that $p_n \to f$ on $\mathbb{R}$, where

   a) $f(x) = \sin x$ b) $f(x) = e^x$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that we can find a sequence of polynomials $\langle p_n \rangle$ such that $p_n \to f$ on any bounded subset of $\mathbb{R}$.

5. Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = \dfrac{1}{x}$. Show that there does not exist a sequence of polynomials $\langle p_n \rangle$ such that $p_n \to f$ on $(0, 1)$.

6. Keep the hypothesis of the Weierstrass approximation theorem, can we find a sequence of polynomials $\langle p_n \rangle$ such that $\sum p_n = f$ on $[0, 1]$?

6 Sequence of Functions in a Compact Set

The following elementary lemma is needed to prove certain important theorems for the sequence of complex functions in a compact set.

LEMMA **1.** Suppose that the sequence

(i) $\langle f_n(x) \rangle$ converges uniformly to $f(x)$ on a compact set $\mathcal{D}$ and $g(x)$ is a continuous function on $\mathcal{D}$. Then $\langle g(x) f_n(x) \rangle$ converges to $g(x)f(x)$ uniformly on $\mathcal{D}$.

(ii) $\langle u_n(x) \rangle$ converges uniformly to $S(x)$ on a compact set $\mathcal{D}$ and $g(x)$ is a continuous function on $\mathcal{D}$. Then $\sum_{n=1}^{\infty} g(x)u_n(x)$ converges to $g(x)S(x)$ uniformly on $\mathcal{D}$.

THEOREM **20.** *Let $\langle f_n(x) \rangle$ be a sequence of differentiable functions on a domain $\mathcal{D}$. If $f_n(x) \to f(x)$ uniformly on every compact subset of $\mathcal{D}$, then, for any $k \geq 1$, $f_n^{(k)}(x) \to f^{(k)}(x)$ for all $x \in \mathcal{D}$; i.e., the limit of the kth derivative is the kth derivative of the limit. Moreover, for each $k \geq 1$, the differentiated sequence $\left\langle f_n^{(k)}(x) \right\rangle$ converges to $f^{(k)}(x)$ uniformly on every compact subset of $\mathcal{D}$.*

Proof:

RESULT **8.** The above Theorem 20 does not hold if $\mathcal{D}$ is assumed to be an arbitrary set instead of a domain. The sequence $\left\{ \frac{\sin nx}{n} \right\}$ converges uniformly to zero on the real axis; however, the sequence of its derivative $\{\cos nx\}$ converges only at $x = 0$. Thus, the sequence $\left\{ \frac{\sin nz}{n} \right\}$ cannot converge uniformly on any domain containing points of the real axis.

Definition 6. A sequence $\langle f_n(x) \rangle$ of differentiable functions of a domain $\mathcal{D} \subseteq \mathbb{C}$ *converges normally* to the differentiable function $f(z)$ on $\mathcal{D}$ if it converges uniformly to $f(z)$ on each compact subsets in $\mathcal{D}$.

6.1 Convergence in the Space of Differentiable Functions

In this section, we shall prove two important theorems namely, Ascoli-Arzela theorem and the Motel's theorem which guarantees that, given a family $\mathcal{F}$ of functions in $\mathbb{R}$ that any sequence in $\mathcal{F}$ have a uniformly convergent subsequence.

Definition 7. [Equicontinuous]: Let $\mathcal{F}$ be a family of collection of functions, defined and continuous and real-valued on a compact subset $E \subset \mathbb{R}$. A function $f \in \mathcal{F}$ is continuous at $x_0 \in E$ if given $\varepsilon > 0$, $\exists \delta = \delta(f, x_0, \varepsilon) > 0$ such that

$$|x - x_0| < \delta \text{ and } x \in E \Rightarrow \left| f(x) - f(x_0) \right| < \varepsilon.$$

If δ, independent of f and depending on x_0 and ε can be found $\forall \varepsilon > 0$, we say that the family $\mathcal{F}$ is *equicontinuous* at x_0.

Definition 8. [Uniformly bounded]: Let $\langle f_n \rangle$ be a sequence of differentiable functions on a domain $\mathcal{D} \subset \mathbb{R}$ and let $U \subset \mathcal{D}$. Then, we say that $\langle f_n \rangle$ is uniformly bounded on U, if $\exists$ a $M > 0$ such that

$$|f(x)| \leq M; \forall x \in \mathcal{D} \text{ and } \forall\, n \in \mathbb{N}.$$

Let $\mathcal{F}$ be a family of collection of functions, defined and continuous and real-valued on a compact subset $E \subset \mathbb{R}$. We say that $\mathcal{F}$ is *uniformly bounded on* E if

$$|f(x)| \leq M; \forall x \in E \text{ and } \forall f \in \mathcal{F}$$

Definition 9. [Locally bounded]: A set $\mathcal{F} \subset \mathcal{H}(\mathcal{D})$ is locally bounded if for each point $\alpha \in \mathcal{D}$, there are constants M and $\rho > 0$ such that

$$|f(x)| \leq M; |x - \alpha| < \rho \text{ for all } f \in \mathcal{F}$$
$$\text{i.e.,} \quad \sup \Big\{ |f(x)| : |x - \alpha| < \rho, f \in \mathcal{F} \Big\} < \infty$$

That is, $\mathcal{F}$ is locally bounded if about a point $\alpha \in \mathcal{D}$, there is a interval on which $\mathcal{F}$ is uniformly bounded, which immediately extends to the requirement that $\mathcal{F}$ is uniformly bounded on a compact sets in $\mathcal{D}$.

Definition 10. [Normal convergence]: If a sequence of real functions $\langle f_n \rangle$ converges on a compact subsets, it is called *normal convergence*. If a sequence of functions is uniformly bounded on compact subsets of the domain, it is said to be *normally bounded.*

THEOREM **21. Ascoli-Arzela Theorem :** *Let $\mathcal{F}$ be a family of collection of continuous, real-valued function on a compact subset $E \subset \mathbb{R}$. Suppose $\mathcal{F}$ is uniformly bounded on E. Then the followings are equivalent:*

(i) $\mathcal{F}$ is equicontinuous at each point of E

(ii) Every sequence $\Big\langle f_n \Big\rangle \subset \mathcal{F}$ has a uniformly convergent subsequence $\Big\langle f_{n_k} \Big\rangle$

Proof: Let $E_{\mathbb{Q}}$ be the set of points of E with rational coordinates as

$$E_{\mathbb{Q}} = \Big\{ r_n : r_n \in \mathbb{Q} \Big\} = E \cap \mathbb{Q}$$

Since $\mathbb{Q}$ is countable and dense subset in E, then $E_{\mathbb{Q}}$ is countable and $\overline{E}_{\mathbb{Q}} = E$. As $E_{\mathbb{Q}}$ is countable, we may enumerate the points of $E_{\mathbb{Q}}$ by $r_1, r_2, \cdots$, i.e., $E_{\mathbb{Q}} = \Big\langle r_n \Big\rangle_{n \geq 1}$.

Let $\Big\langle f_n \Big\rangle$ be a sequence of $\mathcal{F}$. Since $\mathcal{F}$ is uniformly bounded on each compact subset of E and hence it is pointwise bounded. Thus, if we consider $\Big\langle f_n(r_1) \Big\rangle_{n \geq 1}$, then it is a bounded sequence of real numbers, and so $|f_n(r_1)| \leq M$. By Bolzano-Weierstrass theorem, we have a convergent subsequence $\Big\langle f_{n_1}(r_1) \Big\rangle_{n_1 \geq 1}$ of $\Big\langle f_n(r_1) \Big\rangle$.

Now, consider $\Big\langle f_{n_1}(r_2) \Big\rangle_{n_1 \geq 1}$. Again $\Big| f_{n_1}(r_2) \Big| \leq M$ and so by Bolzano-Weierstrass theorem, $\exists$ a subsequence $\Big\langle f_{n_2}(r_2) \Big\rangle$ of $\Big\langle f_{n_1}(r_2) \Big\rangle_{n_1 \geq 1}$ which converges. Note that $\Big\langle f_{n_2}(r_1) \Big\rangle$ also converges.

By induction, we get for $\forall n \geq 1$, a subsequence $\langle f_{n_k} \rangle$ of $\langle f_{n_{k-1}} \rangle$ such that $\Big\langle f_{n_k}(r_j) \Big\rangle$ converges for $j \leq k$, and

$$\langle f_{n_1} \rangle \supseteq \langle f_{n_2} \rangle \supseteq \cdots \supseteq \langle f_{n_{k-1}} \rangle \supseteq \langle f_{n_k} \rangle \supseteq \cdots$$

We get a list of lists:

					Converge at
$f_{n_1}:$	$f_{i_1}(r_1) \searrow$	$f_{i_2}(r_1)$	$f_{i_3}(r_1)$	$\cdots$	r_1
$f_{n_2}:$	$f_{j_1}(r_2)$	$f_{j_2}(r_2) \searrow$	$f_{j_3}(r_3)$	$\cdots$	r_1, r_2
$f_{n_3}:$	$f_{k_1}(r_3)$	$f_{k_2}(r_3)$	$f_{k_3}(r_3) \searrow$	$\cdots$	r_1, r_2, r_3
$\vdots$	$\vdots$			$\cdots$	$\vdots$

Consider the subsequence $\langle F_l \rangle$, where $F_l = l$th member of $\Big\langle f_{n_l} \Big\rangle$, then $\langle F_l \rangle \subset \bigcap\limits_{k=1}^{\infty} \Big\langle f_{n_k} \Big\rangle$, where $\Big\langle f_{n_l} \Big\rangle$ converge at the points $r_1, r_2, r_3, \cdots$. If $x \in E_{\mathbb{Q}}$, then at $x = r_l$ and since $\Big\langle f_{n_l}(r_l) \Big\rangle$ converges we have, the sequence $\Big\langle F_t(r_l) \Big\rangle_{t \geq l}$ converges.

Since $\mathcal{F}$ is equicontinuous on E, so is $\langle F_l \rangle_{l \geq 1}$. So given $x_0 \in E$, and an $\varepsilon > 0$, $\exists \delta > 0$ with $\delta = \delta(x_0, \varepsilon)$ such that

$$\Big| f(x) - f(x_0) \Big| < \frac{\varepsilon}{3}, \text{ whenever } |x - x_0| < \delta, \forall f \in \mathcal{F} \tag{16}$$

The collection $\Big\{ |x - x_0| < \dfrac{\delta(x_0, \varepsilon)}{2} : x_0 \in E \Big\}$ is an open cover of E, which is compact, so admits a finite subcover:

$$\Big\{ |x - \zeta_1| < \frac{\delta}{2}(\zeta_1, \varepsilon); |x - \zeta_2| < \frac{\delta}{2}(\zeta_2, \varepsilon); \cdots, |x - \zeta_k| < \frac{\delta}{2}(\zeta_k, \varepsilon) \Big\}$$

Choose $x_{i_1} \in \Big\{ |x - \zeta_1| < \frac{\delta}{2}(\zeta_1, \varepsilon) \Big\}$; $x_{i_2} \in \Big\{ |x - \zeta_2| < \frac{\delta}{2}(\zeta_2, \varepsilon) \Big\} \cdots$; $x_{i_k} \in \Big\{ |x - \zeta_k| < \frac{\delta}{2}(\zeta_k, \varepsilon) \Big\}$ where, $x_{i_j} \in E_{\mathbb{Q}} = \Big\langle r_n \Big\rangle_{n \geq 1}$. Note that

$$\begin{aligned} & |x - x_{i_l}| \leq |x - \zeta_{i_l}| + |\zeta_{i_l} - x_{i_l}| < \delta(\zeta_l, \varepsilon) \\ \Rightarrow \quad & \Big| F_m(x) - F_m(x_{i_l}) \Big| < \frac{\varepsilon}{3} \text{ and } \Big| F_n(x_{i_l}) - F_n(x) \Big| < \frac{\varepsilon}{3} \end{aligned}$$

If $x \in E$, then $|x - \zeta_l| < \delta(\zeta_l, \varepsilon)/2$ for some l with $1 \leq l \leq k$ and x_{i_l} is also in $|x - \zeta_k| < \delta(\zeta_k, \varepsilon)/2$. Let m, n be two positive integers, greater than a large positive integer. Now

$$\begin{aligned} |F_m(x) - F_n(x)| &= \Big| F_m(x) - F_m(x_{i_l}) + F_m(x_{i_l}) - F_n(x_{i_l}) + F_n(x_{i_l}) - F_n(x) \Big| \\ &\leq \Big| F_m(x) - F_m(x_{i_l}) \Big| + \Big| F_m(x_{i_l}) - F_n(x_{i_l}) \Big| + \Big| F_n(x_{i_l}) - F_n(x) \Big| \end{aligned}$$

Since $\langle F_m \rangle$ converge on $E_{\mathbb{Q}}$ and $x_{i_l} \in E_{\mathbb{Q}}$, for the given ε, $\exists \mathcal{N}(\varepsilon, x_{i_l})$ such that

$$\Big| F_m(x_{i_l}) - F_n(x_{i_l}) \Big| < \frac{\varepsilon}{3} \text{ for } m, n \geq \mathcal{N}(\varepsilon, x_{i_l})$$

Thus for $m, n \geq \max\left\{\mathcal{N}(\varepsilon, x_{i_l}) : 1 \leq l \leq k\right\}$ we have

$$\begin{aligned} & |F_m(x) - F_n(x)| < \varepsilon \text{ independent of } x \in E \\ \Rightarrow \quad & \langle F_m \rangle \text{ converges uniformly on } E. \end{aligned}$$

This proves the theorem. □

THEOREM **22.** *Let $f_n : [a, b] \to \mathbb{R}$ be continuous such that $\langle f_n \rangle$ are uniformly bounded on $[a, b]$ and the derivatives f_n' exist and are uniformly bounded on $(a, b0$. Then f_n has a uniformly convergent subsequence.*

Proof: Since f_n' are uniformly bounded on (a, b), there exists $M > 0$ such that $|f_n'(x)| \leq M$, $\forall n \in \mathbb{N}$ and for any $x \in (0, 1)$. Using the mean value theorem, we get

$$|f_n(x) - f_n(y)| \leq M|x - y|; \text{ for any } x, y \in [a, b] \text{ and } n \in \mathbb{N}$$

So, if $\varepsilon > 0$ is given, set $\delta = \varepsilon/(M + 1)$. Then

$$|f_n(x) - f_n(y)| \leq M|x - y|; \text{ for any } n \in \mathbb{N} \text{ with } |x - y| < \delta$$

This shows that $\langle f_n \rangle$ is equicontinuous. Let us prove that $\langle f_n \rangle$ has a subsequence which converges uniformly. The Arzela-Ascoli theorem then supplies the uniformly convergent subsequence.

EXAMPLE **58.** *Which of the following sequences $\langle f_n \rangle$ in $\mathcal{C}[0, 1]$ must contain a uniformly convergent subsequence?*

a) When $\{f_n(x)\} \leq 3$ for all $x \in [0, 1]$ and for all $n \in \mathbb{N}$

b) When $f_n \in \mathcal{C}^1[0, 1]$, $|f_n(x)| \leq 3$ and $|f_n'(x)| \leq 5$ for all $x \in [0, 1]$ and for all $n \in \mathbb{N}$

Solution: (a) Let $f_n(x) = x^n$, for $\mathcal{C}$, implies $||f_n|| = 1$ for all $n \in \mathbb{N}$. To prove that there is no convergent subsequence for this sequence, it is sufficient to show that any subsequence of $\langle f_n \rangle$ is not Cauchy. (Since every convergent sequence is a Cauchy sequence). Observe that:

$$||f_{2n} - f_n|| = \sup_{x \in [0,1]} (x^n - x^{2n}) = \sup_{x \in [0,1]} (x^n - (x^n)^2) = \sup_{t \in [0,1]} (t - t^2) = \frac{1}{4}$$

Since $\langle f_n \rangle$ is monotonic, we see that

$$k \geq 2n \Rightarrow ||f_k - f_n|| \geq \frac{1}{4}.$$

Now, we have any subsequence of $\langle f_n \rangle$ then the above estimate shows that this subsequence is not Cauchy. For any given k_0, we can find $k' > k_0$ such that $n_{k'} > 2n_{k_0}$ and $||f_{n_{k'}} - f_{n_{k_0}}|| \geq \frac{1}{4}$. So this option is incorrect.
(b) Let $\langle f_n \rangle$ be a uniformly bounded sequence of real-valued differentiable functions on $[a, b]$ such that the derivatives $\langle f_n' \rangle$ are uniformly bounded. Then by Arzela-Ascoli theorem, we conclude that there exists a subsequence of $\langle f_n \rangle$ that converges uniformly on $[a, b]$. Thus this option is correct.

Problem Set

Short answer type questions

1. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and the sequence $f_n(x) = f(nx)$ is equicontinuous, what can be said about f?

Long answer type questions

1. Is the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \cos(n+x) + \log\left(1 + \frac{1}{\sqrt{n+2}} \sin^2(n^n x)\right)$$

equicontinuous? Prove or disprove.

2. Let $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\int_{-\infty}^{\infty} |f(x)|dx < \infty$. Show that there is a sequence $\langle x_n \rangle$ in $\mathbb{R}$ such that $x_n \to \infty$, $x_n f(x_n) \to 0$, and $x_n f(-x_n) \to 0$ as $n \to \infty$.

3. Let $\langle f_n \rangle$ be a sequence of functions are continuous over $[0,1]$ and continuously differentiable in $(0,1)$. Assume that $|f_n(x)| \leq 1$ and that $|f_n'(x)| \leq 1$ for all $x \in (0,1)$ and for each $n \in \mathbb{N}$. Pick the true statements: **NBHM'09**

   a) f_n is uniformly continuous for each n **Hints :** Since f_n is continuous on a compact set $[0,1]$

   b) $\langle f_n \rangle$ contain a subsequence which converges in $\mathcal{C}[0,1]$ **Hints :** Arzela-Ascoli theorem

   c) $\langle f_n \rangle$ is a convergent sequence in $\mathcal{C}[0,1]$. **Hints :** $f_n(x) = (-1)^n$

   Ans: 3 (a), (b)

4. Which of the following sequences $\langle f_n \rangle$ in $\mathcal{C}[0,1]$ must contain a uniformly convergent subsequence? **NBHM'15**

   a) When $\{f_n(x)\} \leq 3$ for all $x \in [0,1]$ and for all $n \in \mathbb{N}$

   b) When $f_n \in \mathcal{C}^1[0,1]$, $|f_n(x)| \leq 3$ and $|f_n'(x)| \leq 5$ for all $x \in [0,1]$ and for all $n \in \mathbb{N}$

   c) When $f_n \in \mathcal{C}^1[0,1]$ and $\int_0^1 |f_n(x)|dx \leq 1$ for all $n \in \mathbb{N}$ **Hints** Take the same example 58 as taken in option (a)

   Ans: 4 (b)

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